doi: 10.17951/a.2020.74.1.1-14

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXIV, NO. 1, 2020 SECTIO A 1–14

DOROTA BRÓD

On split r-Jacobsthal quaternions

Abstract. In this paper we introduce a one-parameter generalization of the split Jacobsthal quaternions, namely the split r-Jacobsthal quaternions. We give a generating function, Binet formula for these numbers. Moreover, we obtain some identities, among others Catalan, Cassini identities and convolution identity for the split r-Jacobsthal quaternions.

1. Introduction. A quaternion p is a hyper-complex number represented by an equation

$$p = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where $a, b, c, d \in \mathbb{R}$ and $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal basis in $\mathbb{R}^4$, which satisfies the quaternion multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The quaternions were discovered in 1843 by W. R. Hamilton. In 1849 ([3]), J. Cockle introduced split quaternions, which were called coquaternions. A split quaternion q with real components a_0, a_1, a_2, a_3 and basis $\{1, i, j, k\}$ has the form

$$q = a_0 + a_1 i + a_2 j + a_3 k, \tag{1}$$

where the imaginary units satisfy the non-commutative multiplication rules:

$$i^2 = -1,\ j^2 = k^2 = ijk = 1, \tag{2}$$

2010 *Mathematics Subject Classification.* 11B37, 11R52.

Key words and phrases. Jacobsthal numbers, quaternion, split quaternion, split Jacobsthal quaternion, Binet formula.

(3) $$ij = k = -ji,\ jk = -i = -kj,\ ki = j = -ik.$$

The scalar and vector parts of a split quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ are denoted by $S_q = a_0$, $\vec{V}_q = a_1 i + a_2 j + a_3 k$, respectively. Hence we get $q = S_q + \vec{V}_q$. The conjugate of the split quaternion denoted by $\overline{q}$, is given by

$$\overline{q} = a_0 - a_1 i - a_2 j - a_3 k.$$

The norm of q is defined as

(4) $$N(q) = q\overline{q} = a_0^2 + a_1^2 - a_2^2 - a_3^2.$$

The split quaternions are elements of a 4-dimensional associative algebra. They form a 4-dimensional real vector space equipped with a multiplicative operation. The split quaternions contain nontrivial zero divisors, nilpotent elements and idempotents, for example $\frac{1+j}{2}$ is an idempotent zero divisor, and $i - j$ is nilpotent.

Let q_1, q_2 be any two split quaternions, $q_1 = a_0 + a_1 i + a_2 j + a_3 k$, $q_2 = b_0 + b_1 i + b_2 j + b_3 k$. Then addition and subtraction of the split quaternions are defined as follows:

$$q_1 \pm q_2 = (a_0 \pm b_0) + (a_1 \pm b_1)i + (a_2 \pm b_2)j + (a_3 \pm b_3)k.$$

Multiplication of the split quaternions is defined by

$$\begin{aligned} q_1 \cdot q_2 = {} & (a_0b_0 - a_1b_1 + a_2b_2 + a_3b_3) + (a_0b_1 + a_1b_0 - a_2b_3 + a_3b_2)i \\ & + (a_0b_2 + a_2b_0 - a_1b_3 + a_3b_1)j + (a_0b_3 + a_3b_0 - a_2b_1 + a_1b_2)k. \end{aligned}$$

2. The r-Jacobsthal numbers. In [6], A. F. Horadam introduced a second order linear recurrence sequence $\{w_n\}$ by the relations

(5) $$w_0 = a,\ w_1 = b,\ w_n = pw_{n-1} - qw_{n-2}$$

for $n \geq 2$ and arbitrary integers a, b, p, q. This sequence is a certain generalization of famous sequences such as Fibonacci sequence ($a = 0$, $b = 1$, $p = 1$, $q = -1$), Lucas sequence ($a = 2$, $b = 1$, $p = 1$, $q = -1$), Pell sequence ($a = 0$, $b = 1$, $p = 2$, $q = -1$). Hence sequences defined by (5) are called sequences of the Fibonacci type. Numbers of the Fibonacci type appear in many subjects of mathematics. In [7], A. F. Horadam defined the Fibonacci and Lucas quaternions. In [1], the split Fibonacci quaternions Q_n and the split Lucas quaternions T_n were introduced as follows:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where F_n is the nth Fibonacci number, L_n is the nth Lucas number and i, j, k are split quaternions units which satisfy the rules (2) and (3).

A generalization of the split Fibonacci quaternions split k-Fibonacci quaternions was investigated in [9]. The authors used a generalization of the Fibonacci numbers and the Lucas numbers: k-Fibonacci numbers and

k-Lucas numbers. Some interesting results for the split Pell quaternions and the split Pell–Lucas quaternions can be found in [10]. In [11], the split Jacobsthal quaternions and the split Jacobsthal–Lucas quaternions were considered.

The Jacobsthal sequence $\{J_n\}$ is defined by the recurrence

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2 \tag{6}$$

with initial conditions $J_0 = 0$, $J_1 = 1$. The first ten terms of the sequence are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171$. This sequence is also given by the Binet-type formula

$$J_n = \frac{2^n - (-1)^n}{3} \text{ for } n \geq 0.$$

Many authors introduced and studied some generalizations of the recurrence of the Jacobsthal sequence, see [4, 5]. The second order recurrence (6) has been generalized in two ways: first, by preserving the initial conditions and second, by preserving the recurrence relation. In [2], a one-parameter generalization of the Jacobsthal numbers was introduced. We recall this generalization.

Let $n \geq 0$, $r \geq 0$ be integers. The nth r-Jacobsthal number $J(r, n)$ is defined as follows:

$$J(r,n) = 2^r J(r, n-1) + (2^r + 4^r) J(r, n-2) \text{ for } n \geq 2 \tag{7}$$

with $J(r, 0) = 1$, $J(r, 1) = 1 + 2^{r+1}$.

For $r = 0$ we have $J(0, n) = J_{n+2}$. By (7) we obtain

$$\begin{aligned}
J(r,0) &= 1\\
J(r,1) &= 2 \cdot 2^r + 1\\
J(r,2) &= 3 \cdot 4^r + 2 \cdot 2^r\\
J(r,3) &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r\\
J(r,4) &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r\\
J(r,5) &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r.
\end{aligned} \tag{8}$$

In [2], it was proved that the r-Jacobsthal numbers can be used for counting of independent sets of special classes of graphs. We will recall some properties of the r-Jacobsthal numbers.

Theorem 1 ([2], Binet formula). *For $n \geq 0$, the nth r-Jacobsthal number is given by*

$$J(r,n) = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} {\lambda_1}^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} {\lambda_2}^n,$$

where

$$\lambda_1 = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \lambda_2 = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Theorem 2 ([2]). *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$\sum_{l=0}^{n-1} J(r,l) = \frac{J(r,n) + (2^r + 4^r)J(r,n-1) - 2 - 2^r}{4^r + 2^{r+1} - 1}. \tag{9}$$

Theorem 3 ([2], Cassini identity). *Let $n \geq 1$. Then*

$$J(r,n+1)J(r,n-1) - J^2(r,n) = (-1)^n(2^r+1)^2(2^r+4^r)^{n-1}.$$

Theorem 4 ([2], convolution identity). *Let n, m, r be integers such that $m \geq 2$, $n \geq 1$, $r \geq 0$. Then*

$$J(r,m+n) = 2^r J(r,m-1)J(r,n) + (4^r + 8^r)J(r,m-2)J(r,n-1).$$

In this paper, we introduce and study split r-Jacobsthal quaternions. Another generalization of the split Jacobsthal quaternions was studied in [8].

3. Some properties of the split r-Jacobsthal quaternions. For $n \geq 0$, the split r-Jacobsthal quaternion JSQ_n^r we define by

$$JSQ_n^r = J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3), \tag{10}$$

where $J(r,n)$ is the nth r-Jacobsthal number, defined by (7) and i, j, k are split quaternions units which satisfy the multiplication rules (2) and (3).

By (8) and (10) we obtain

$$\begin{aligned} JSQ_0^r &= 1 + i(2^{r+1}+1) + j(3\cdot 4^r + 2^{r+1}) + k(5\cdot 8^r + 5\cdot 4^r + 2^r) \\ JSQ_1^r &= 2^{r+1} + 1 + i(3\cdot 4^r + 2^{r+1}) + j(5\cdot 8^r + 5\cdot 4^r + 2^r) \\ &\quad + k(8\cdot 16^r + 10\cdot 8^r + 3\cdot 4^r) \\ JSQ_2^r &= 3\cdot 4^r + 2^{r+1} + i(5\cdot 8^r + 5\cdot 4^r + 2^r) \\ &\quad + j(8\cdot 16^r + 10\cdot 8^r + 3\cdot 4^r) \\ &\quad + k(13\cdot 32^r + 20\cdot 16^r + 9\cdot 8^r + 4^r). \end{aligned} \tag{11}$$

Using the formula $J(0,n) = J_{n+2}$, we obtain $JSQ_n^0 = JSQ_{n+2}$, where JSQ_n is the nth split Jacobsthal quaternion introduced in [11].

Proposition 5. *Let $n \geq 0$, $r \geq 0$. Then*

$$\begin{aligned} N(JSQ_n^r) &= (1 - 4^r - 2\cdot 8^r - 2\cdot 16^r - 2\cdot 32^r - 64^r)J^2(r,n) \\ &\quad + (1 - 2\cdot 4^r - 4\cdot 8^r - 4\cdot 16^r)J^2(r,n+1) \\ &\quad - 2(4^r + 2\cdot 8^r + 3\cdot 16^r + 2\cdot 32^r)J(r,n)J(r,n+1). \end{aligned}$$

Proof. By (7) we get

$$\begin{aligned} J(r,n+2) &= 2^r J(r,n+1) + (2^r + 4^r)J(r,n), \\ J(r,n+3) &= (2^r + 2\cdot 4^r)J(r,n+1) + (4^r + 8^r)J(r,n). \end{aligned}$$

Let $A = J(r, n+1)$, $B = J(r, n)$. Using formula (4), we obtain

$$\begin{aligned} N(JSQ_n^r) &= A^2 + B^2 - (2^r A + (2^r + 4^r)B)^2 - ((2^r + 2\cdot 4^r)A \\ &\quad + (4^r + 8^r)B)^2 \\ &= \left[1 - 4^r - (2\cdot 4^r + 2^r)^2\right]A^2 + \left[1 - (2^r + 4^r)^2 - (4^r + 8^r)^2\right]B^2 \\ &\quad - 2[4^r + 8^r + (2\cdot 4^r + 2^r)(4^r + 8^r)]AB. \end{aligned}$$

By simple calculations we get the result. □

Proposition 6. *Let $n \geq 2$, $r \geq 0$. Then*

$$JSQ_n^r = 2^r JSQ_{n-1}^r + (2^r + 4^r)JSQ_{n-2}^r,$$

where JSQ_0^r, JSQ_1^r are given in (11).

Proof. By (10) we get

$$\begin{aligned} &2^r JSQ_{n-1}^r + (2^r + 4^r)JSQ_{n-2}^r \\ &\quad = 2^r(J(r, n-1) + iJ(r, n) + jJ(r, n+1) + kJ(r, n+2)) \\ &\qquad + (2^r + 4^r)(J(r, n-2) + iJ(r, n-1) + jJ(r, n) + kJ(r, n+1)) \\ &\quad = J(r, n) + iJ(r, n+1) + jJ(r, n+2) + kJ(r, n+3) = JSQ_n^r. \end{aligned}$$ □

Proposition 7. *Let $n \geq 0$, $r \geq 0$. Then*

(i) $JSQ_n^r + \overline{JSQ_n^r} = 2J(r, n)$,
(ii) $(JSQ_n^r)^2 = 2J(r, n)JSQ_n^r - N(JSQ_n^r)$,
(iii) $JSQ_n^r - iJSQ_{n+1}^r - jJSQ_{n+2}^r - kJSQ_{n+3}^r$
$= J(r, n) + J(r, n+2) - J(r, n+4) - J(r, n+6)$.

Proof. (i) By the definition of the conjugate of the split quaternion we have

$$\begin{aligned} JSQ_n^r + \overline{JSQ_n^r} &= J(r, n) + iJ(r, n+1) + jJ(r, n+2) + kJ(r, n+3) \\ &\quad + J(r, n) - iJ(r, n+1) - jJ(r, n+2) - kJ(r, n+3) \\ &= 2J(r, n). \end{aligned}$$

(ii) By simple calculations we obtain

$$\begin{aligned} (JSQ_n^r)^2 &= J^2(r, n) - J^2(r, n+1) + J^2(r, n+2) + J^2(r, n+3) \\ &\quad + 2(iJ(r, n)J(r, n+1) + jJ(r, n)J(r, n+2) + J(r, n)J(r, n+3)) \\ &\quad + (ij + ji)J(r, n+1)J(r, n+2) + (ik + ki)J(r, n+1)J(r, n+3) \\ &\quad + (jk + kj)J(r, n+2)J(r, n+3). \end{aligned}$$

By (3) we get

$$\begin{aligned}(JSQ_n^r)^2 &= -J^2(r,n) - J^2(r,n+1) + J^2(r,n+2) + J^2(r,n+3)\\ &\quad + 2(J^2(r,n) + iJ(r,n)J(r,n+1)\\ &\quad + jJ(r,n)J(r,n+2) + kJ(r,n)J(r,n+3))\\ &= 2J(r,n)(J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3))\\ &\quad - (J^2(r,n) + J^2(r,n+1) - J^2(r,n+2) - J^2(r,n+3))\\ &= 2J(r,n)JSQ_n^r - N\left(JSQ_n^r\right).\end{aligned}$$

(iii)

$$\begin{aligned}&JSQ_n^r - iJSQ_{n+1}^r - jJSQ_{n+2}^r - kJSQ_{n+3}^r\\ &= J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3)\\ &\quad - i(J(r,n+1) + iJ(r,n+2) + jJ(r,n+3) + kJ(r,n+4))\\ &\quad - j(J(r,n+2) + iJ(r,n+3) + jJ(r,n+4) + kJ(r,n+5))\\ &\quad - k(J(r,n+3) + iJ(r,n+4) + jJ(r,n+5) + kJ(r,n+6))\\ &= J(r,n) + J(r,n+2) - J(r,n+4) - J(r,n+6)\\ &\quad - (ij+ji)J(r,n+3) - (ik+ki)J(r,n+4) - (jk+kj)J(r,n+5).\end{aligned}$$

Using equalities $ij + ji = 0$, $ik + ki = 0$ and $jk + kj = 0$, we get

$$\begin{aligned}&JSQ_n^r - iJSQ_{n+1}^r - jJSQ_{n+2}^r - kJSQ_{n+3}^r\\ &\qquad = J(r,n) + J(r,n+2) - J(r,n+4) - J(r,n+6).\end{aligned}$$

□

Now we present the Binet formula for the split r-Jacobsthal quaternions.

Theorem 8 (Binet formula). *Let $n \geq 0$, $r \geq 0$. Then*

$$JSQ_n^r = C_1\underline{\alpha}\alpha^n + C_2\underline{\beta}\beta^n, \tag{12}$$

where

$$\alpha = 2^{r-1} + \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \quad \beta = 2^{r-1} - \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r},$$

$$\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3, \quad \underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3,$$

$$C_1 = \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} + 3\cdot 2^r + 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}, \quad C_2 = \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} - 3\cdot 2^r - 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}.$$

Proof. By the Binet formula for the r-Jacobsthal numbers we obtain

$$\begin{aligned}JSQ_n^r &= J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3)\\ &= C_1\alpha^n + C_2\beta^n + i\big(C_1\alpha^{n+1} + C_2\beta^{n+1}\big)\\ &\quad + j\big(C_1\alpha^{n+2} + C_2\beta^{n+2}\big) + k\big(C_1\alpha^{n+3} + C_2\beta^{n+3}\big)\\ &= C_1\alpha^n\left(1 + i\alpha + j\alpha^2 + k\alpha^3\right) + C_2\beta^n\left(1 + i\beta + j\beta^2 + k\beta^3\right)\\ &= C_1\underline{\alpha}\alpha^n + C_2\underline{\beta}\beta^n.\end{aligned}$$

□

In particular, we obtain the Binet formula for the split Jacobsthal quaternions (see [11]).

Corollary 9. *Let $n \geq 0$ be an integer. Then*

$$JSQ_n = \frac{1}{3}\big[2^n(1+2i+4j+8k) - (-1)^n(1-i+j-k)\big].$$

Proof. By Theorem 8, for $r=0$ we have $C_1 = \frac{4}{3}$, $C_2 = -\frac{1}{3}$, $\alpha = 2$, $\beta = -1$ and

$$\begin{aligned} JSQ_n^0 &= \frac{4}{3}\cdot 2^n(1+2i+4j+8k) - \frac{1}{3}(-1)^n(1-i+j-k) \\ &= \frac{1}{3}\cdot 2^{n+2}(1+2i+4j+8k) - \frac{1}{3}(-1)^{n+2}(1-i+j-k) = JSQ_{n+2}. \square \end{aligned}$$

The Binet formula (12) can be used for proving some identities for the split r-Jacobsthal quaternions. We will need the following lemma.

Lemma 10. *Let*

$$\begin{aligned} \alpha &= 2^{r-1} + \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \\ \beta &= 2^{r-1} - \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \\ \underline{\alpha} &= 1 + i\alpha + j\alpha^2 + k\alpha^3, \\ \underline{\beta} &= 1 + i\beta + j\beta^2 + k\beta^3. \end{aligned}$$

Then

$$\begin{aligned} \underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha} = 2\big[&1 + 4^r + 2^r + (4^r+2^r)^2 - (4^r+2^r)^3 \\ &+ 2^r i + (3\cdot 4^r + 2^{r+1})j + (4\cdot 8^r + 3\cdot 4^r)k\big]. \end{aligned}$$

Proof. By (2) and (3) we have

$$\begin{aligned} \underline{\alpha}\underline{\beta} &= 1 - \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + i\big(\alpha+\beta+(\alpha\beta)^2(\alpha-\beta)\big) \\ &\quad + j\big(\alpha^2+\beta^2+\alpha\beta(\alpha^2-\beta^2)\big) + k\big(\alpha^3+\beta^3+\alpha\beta(\beta-\alpha)\big), \\ \underline{\beta}\underline{\alpha} &= 1 - \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + i\big(\alpha+\beta-(\alpha\beta)^2(\alpha-\beta)\big) \\ &\quad + j\big(\alpha^2+\beta^2-\alpha\beta(\alpha^2-\beta^2)\big) + k\big(\alpha^3+\beta^3-\alpha\beta(\beta-\alpha)\big). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha} = 2\big[&1 - \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + i(\alpha+\beta) \\ &+ j\big(\alpha^2+\beta^2\big) + k\big(\alpha^3+\beta^3\big)\big]. \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta &= 2^r, \\ \alpha - \beta &= \sqrt{4\cdot 2^r + 5\cdot 4^r}, \\ \alpha\beta &= -(4^r + 2^r), \end{aligned}$$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=3\cdot 4^r+2^{r+1},$$
$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=4\cdot 8^r+3\cdot 4^r.$$

Hence we get

$$\begin{aligned}\underline{\alpha}\underline{\beta}+\underline{\beta}\underline{\alpha}=2\big[&1+4^r+2^r+(4^r+2^r)^2-(4^r+2^r)^3\\&+2^ri+(3\cdot 4^r+2^{r+1})j+(4\cdot 8^r+3\cdot 4^r)k\big].\end{aligned}\tag{13}$$

Moreover,

$$\begin{aligned}\underline{\alpha}\underline{\beta}=\;&1+4^r+2^r+(4^r+2^r)^2-(4^r+2^r)^3\\&+i\big(2^r+(4^r+2^r)^2\sqrt{4\cdot 2^r+5\cdot 4^r}\big)\\&+j\big(3\cdot 4^r+2^{r+1}-(8^r+4^r)\sqrt{4\cdot 2^r+5\cdot 4^r}\big)\\&+k\big(4\cdot 8^r+3\cdot 4^r+(4^r+2^r)\sqrt{4\cdot 2^r+5\cdot 4^r}\big),\end{aligned}\tag{14}$$

$$\begin{aligned}\underline{\beta}\underline{\alpha}=\;&1+4^r+2^r+(4^r+2^r)^2-(4^r+2^r)^3\\&+i\big(2^r-(4^r+2^r)^2\sqrt{4\cdot 2^r+5\cdot 4^r}\big)\\&+j\big(3\cdot 4^r+2^{r+1}+(8^r+4^r)\sqrt{4\cdot 2^r+5\cdot 4^r}\big)\\&+k\big(4\cdot 8^r+3\cdot 4^r-(4^r+2^r)\sqrt{4\cdot 2^r+5\cdot 4^r}\big).\end{aligned}\tag{15}$$

□

Now we will give some identities such as Catalan, Cassini and d'Ocagne identities for the split r-Jacobsthal quaternions.

Theorem 11 (Catalan identity). *Let $n\ge 0$, $r\ge 0$ be integers such that $m\ge n$. Then*

$$\begin{aligned}&(JSQ_m^r)^2-JSQ_{m+n}^rJSQ_{m-n}^r\\&\quad=-\frac{(1+2^r)^2(-4^r-2^r)^{m-n}}{4\cdot 2^r+5\cdot 4^r}\left((-4^r-2^r)^n(\underline{\alpha}\underline{\beta}+\underline{\beta}\underline{\alpha})-\alpha^{2n}\underline{\alpha}\underline{\beta}-\beta^{2n}\underline{\beta}\underline{\alpha}\right).\end{aligned}$$

Proof. By formula (12) we get

$$\begin{aligned}(JSQ_m^r)^2&-JSQ_{m+n}^rJSQ_{m-n}^r\\&=(C_1\underline{\alpha}\alpha^m+C_2\underline{\beta}\beta^m)(C_1\underline{\alpha}\alpha^m+C_2\underline{\beta}\beta^m)\\&\quad-(C_1\underline{\alpha}\alpha^{m+n}+C_2\underline{\beta}\beta^{m+n})(C_1\underline{\alpha}\alpha^{m-n}+C_2\underline{\beta}\beta^{m-n})\\&=C_1C_2\big[(\alpha\beta)^m(\underline{\alpha}\underline{\beta}+\underline{\beta}\underline{\alpha})-(\alpha\beta)^{m-n}\big(\alpha^{2n}\underline{\alpha}\underline{\beta}+\beta^{2n}\underline{\beta}\underline{\alpha}\big)\big].\end{aligned}$$

Using the formula $\alpha\beta=-(4^r+2^r)$, we obtain

$$\begin{aligned}&(JSQ_m^r)^2-JSQ_{m+n}^rJSQ_{m-n}^r\\&\qquad=C_1C_2(-4^r-2^r)^{m-n}\left((-4^r-2^r)^n(\underline{\alpha}\underline{\beta}+\underline{\beta}\underline{\alpha})-\alpha^{2n}\underline{\alpha}\underline{\beta}-\beta^{2n}\underline{\beta}\underline{\alpha}\right),\end{aligned}$$

where

$$C_1C_2=-\frac{(1+2^r)^2}{4\cdot 2^r+5\cdot 4^r}$$

and $\underline{\alpha}\underline{\beta}+\underline{\beta}\underline{\alpha}$, $\underline{\alpha}\underline{\beta}$, $\underline{\beta}\underline{\alpha}$ are given by (13), (14), (15), respectively. □

Note that for $n=1$ we get the Cassini identity for the split r-Jacobsthal quaternions.

Corollary 12. *For $m \geq 1$, $r \geq 0$ we have*

$$(JSQ_m^r)^2 - JSQ_{m+1}^r JSQ_{m-1}^r$$
$$= -\frac{(1+2^r)^2(-4^r-2^r)^{m-1}}{4\cdot 2^r + 5\cdot 4^r}\left(-(4^r+2^r)(\underline{\alpha}\underline{\beta}+\underline{\beta}\underline{\alpha}) - \alpha^2\underline{\alpha}\underline{\beta} - \beta^2\underline{\beta}\underline{\alpha}\right).$$

In particular, we obtain the Cassini identity for the split Jacobsthal quaternions (see [11]).

Corollary 13. *Let $m \geq 1$ be an integer. Then*

$$(JSQ_m)^2 - JSQ_{m+1}JSQ_{m-1} = (-2)^{m-1}(-1+5i+3j+9k).$$

Proof. By (14) and (15) for $r=0$ we have

$$\underline{\alpha}\underline{\beta} = -1+13i-j+13k,$$
$$\underline{\beta}\underline{\alpha} = -1-11i+11j+k.$$

By Corollary 12 we get

$$\begin{aligned}(JSQ_m^0)^2 &- JSQ_{m+1}^0 JSQ_{m-1}^0\\ &= -\frac{4(-2)^{m-1}}{9}\big(-2(-2+2i+10j+14k)\\ &\quad -4(-1+13i-j+13k)-(-1-11i+11j+k)\big)\\ &= 4(-2)^{m-1}(-1+5i+3j+9k).\end{aligned}$$

Using the formula $JSQ_m^0 = JSQ_{m+2}$, we get the result. □

Theorem 14 (d'Ocagne identity). *Let m, n, r be integers. Then*

$$JSQ_n^r JSQ_{m+1}^r - JSQ_{n+1}^r JSQ_m^r$$
$$= \frac{(1+2^r)^2\sqrt{4\cdot 2^r+5\cdot 4^r}}{4\cdot 2^r+5\cdot 4^r}(-4^r-2^r)^m\left(\alpha^{n-m}\underline{\alpha}\underline{\beta} - \beta^{n-m}\underline{\beta}\underline{\alpha}\right),$$

where $\underline{\alpha}\underline{\beta}$, $\underline{\beta}\underline{\alpha}$ are given by (14), (15), *respectively.*

Proof. By formula (12) we get

$$\begin{aligned}JSQ_n^r JSQ_{m+1}^r &- JSQ_{n+1}^r JSQ_m^r\\ &= (C_1\underline{\alpha}\alpha^n + C_2\underline{\beta}\beta^n)(C_1\underline{\alpha}\alpha^{m+1}+C_2\underline{\beta}\beta^{m+1})\\ &\quad -(C_1\underline{\alpha}\alpha^{n+1}+C_2\underline{\beta}\beta^{n+1})(C_1\underline{\alpha}\alpha^m + C_2\underline{\beta}\beta^m)\\ &= C_1C_2(\beta-\alpha)\left(\alpha^n\beta^m\underline{\alpha}\underline{\beta} - \alpha^m\beta^n\underline{\beta}\underline{\alpha}\right)\\ &= C_1C_2(\beta-\alpha)(\alpha\beta)^m\left(\alpha^{n-m}\underline{\alpha}\underline{\beta} - \beta^{n-m}\underline{\beta}\underline{\alpha}\right)\\ &= \frac{(1+2^r)^2\sqrt{4\cdot 2^r+5\cdot 4^r}}{4\cdot 2^r+5\cdot 4^r}(-4^r-2^r)^m\left(\alpha^{n-m}\underline{\alpha}\underline{\beta} - \beta^{n-m}\underline{\beta}\underline{\alpha}\right).\end{aligned}$$

□

In the next theorem we give a summation formula for the split r-Jacobsthal quaternions.

Theorem 15. *Let $n \geq 1$, $r \geq 0$. Then*

$$\sum_{l=0}^{n} JSQ_l^r = \frac{JSQ_{n+1}^r + (2^r + 4^r)JSQ_n^r - (1+i+j+k)(2+2^r)}{4^r + 2^{r+1} - 1} - i - j(2+2^{r+1}) - k(2^{r+2} + 3 \cdot 4^r + 2).$$

Proof. Using Theorem 2, we get

$$\begin{aligned}\sum_{l=0}^{n} JSQ_l^r &= \sum_{l=0}^{n}(J(r,l) + iJ(r,l+1) + jJ(r,l+2) + kJ(r,l+3)) \\ &= \sum_{l=0}^{n} J(r,l) + i\sum_{l=0}^{n} J(r,l+1) + j\sum_{l=0}^{n} J(r,l+2) + k\sum_{l=0}^{n} J(r,l+3) \\ &= \frac{1}{4^r + 2^{r+1} - 1}[J(r,n+1) + (2^r + 4^r)J(r,n) - 2 - 2^r \\ &\quad + i(J(r,n+2) + (2^r + 4^r)J(r,n+1) - 2 - 2^r - J(r,0)) \\ &\quad + j(J(r,n+3) + (2^r + 4^r)J(r,n+2) - 2 - 2^r - J(r,0) - J(r,1)) \\ &\quad + k(J(r,n+4) + (2^r + 4^r)J(r,n+3) - 2 - 2^r \\ &\quad - J(r,0) - J(r,1) - J(r,2))].\end{aligned}$$

By simple calculations we obtain

$$\begin{aligned}\sum_{l=0}^{n} JSQ_l^r &= \frac{1}{4^r + 2^{r+1} - 1}[J(r,n+1) + iJ(r,n+2) \\ &\quad + jJ(r,n+3) + kJ(r,n+4) \\ &\quad + (2^r + 4^r)(J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3)) \\ &\quad - (2+2^r)(1+i+j+k)] - i - j(2^{r+1} + 2) - k(2^{r+2} + 3 \cdot 4^r + 2) \\ &= \frac{JSQ_{n+1}^r + (2^r + 4^r)JSQ_n^r - (1+i+j+k)(2+2^r)}{4^r + 2^{r+1} - 1} \\ &\quad - i - j(2+2^{r+1}) - k(2^{r+2} + 3 \cdot 4^r + 2).\end{aligned}$$

□

The next theorem gives the convolution identity for the split r-Jacobsthal quaternions.

Theorem 16. *Let $m \geq 2$, $n \geq 1$, $r \geq 0$. Then*

$$\begin{aligned}2JSQ_{m+n}^r &= 2^r JSQ_{m-1}^r JSQ_n^r + (4^r + 8^r)JSQ_{m-2}^r JSQ_{n-1}^r \\ &\quad + J(r,m+n) + J(r,m+n+2) - J(r,m+n+4) - J(r,m+n+6).\end{aligned}$$

Proof. By simple calculations we have

$$
\begin{aligned}
&2^r JSQ^r_{m-1} JSQ^r_n \\
&\quad = 2^r(J(r,m-1)J(r,n) + iJ(r,m-1)J(r,n+1) \\
&\quad + jJ(r,m-1)J(r,n+2) + kJ(r,m-1)J(r,n+3) \\
&\quad + iJ(r,m)J(r,n) - J(r,m)J(r,n+1) + kJ(r,m)J(r,n+2) \\
&\quad - jJ(r,m)J(r,n+3) + jJ(r,m+1)J(r,n) - kJ(r,m+1)J(r,n+1) \\
&\quad + J(r,m+1)J(r,n+2) - iJ(r,m+1)J(r,n+3) + kJ(r,m+2)J(r,n) \\
&\quad + jJ(r,m+2)J(r,n+1) + iJ(r,m+2)J(r,n+2) \\
&\quad + J(r,m+2)J(r,n+3)).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&(4^r + 8^r) JSQ^r_{m-2} JSQ^r_{n-1} \\
&\quad = (4^r + 8^r)(J(r,m-2)J(r,n-1) + iJ(r,m-2)J(r,n) \\
&\quad + jJ(r,m-2)J(r,n+1) + kJ(r,m-2)J(r,n+2) \\
&\quad + iJ(r,m-1)J(r,n-1) \\
&\quad - J(r,m-1)J(r,n) + kJ(r,m-1)J(r,n+1) - jJ(r,m-1)J(r,n+2) \\
&\quad + jJ(r,m)J(r,n-1) - kJ(r,m)J(r,n) + J(r,m)J(r,n+1) \\
&\quad - iJ(r,m)J(r,n+2) + kJ(r,m+1)J(r,n-1) + jJ(r,m+1)J(r,n) \\
&\quad + iJ(r,m+1)J(r,n+1) + J(r,m+1)J(r,n+2)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&2^r JSQ^r_{m-1} JSQ^r_n + (4^r + 8^r) JSQ^r_{m-2} JSQ^r_{n-1} \\
&\quad = 2^r J(r,m-1)J(r,n) + (4^r + 8^r)(J(r,m-2)J(r,n-1) \\
&\qquad + i(2^r J(r,m-1)J(r,n+1) + (4^r + 8^r)J(r,m-2)J(r,n)) \\
&\qquad + j(2^r J(r,m-1)J(r,n+2) + (4^r + 8^r)J(r,m-2)J(r,n+1)) \\
&\qquad + k(2^r J(r,m-1)J(r,n+3) + (4^r + 8^r)J(r,m-2)J(r,n+2)) \\
&\qquad + i(2^r J(r,m)J(r,n) + (4^r + 8^r)J(r,m-1)J(r,n-1)) \\
&\qquad + j(2^r J(r,m+1)J(r,n) + (4^r + 8^r)J(r,m)J(r,n-1)) \\
&\qquad + k(2^r J(r,m)J(r,n+2) + (4^r + 8^r)J(r,m-1)J(r,n+1)) \\
&\qquad - 2^r J(r,m)J(r,n+1) - (4^r + 8^r)J(r,m-1)J(r,n) \\
&\qquad + 2^r J(r,m+1)J(r,n+2) - (4^r + 8^r)J(r,m)J(r,n+1) \\
&\qquad + 2^r J(r,m+2)J(r,n+3) - (4^r + 8^r)J(r,m+1)J(r,n+2) \\
&\qquad + i[2^r J(r,m+2)J(r,n+2) + (4^r + 8^r)J(r,m+1)J(r,n+1) \\
&\qquad\quad - 2^r J(r,m+1)J(r,n+3) - (4^r + 8^r)J(r,m)J(r,n+2)]
\end{aligned}
$$

$$\begin{aligned}&+j[2^rJ(r,m+2)J(r,n+1)+(4^r+8^r)J(r,m+1)J(r,n)\\&\quad-2^rJ(r,m)J(r,n+3)-(4^r+8^r)J(r,m-1)J(r,n+2)]\\&+k[2^rJ(r,m+2)J(r,n)+(4^r+8^r)J(r,m+1)J(r,n-1)\\&\quad-2^rJ(r,m+1)J(r,n+1)-(4^r+8^r)J(r,m)J(r,n)].\end{aligned}$$

Using Theorem 4, we get

$$\begin{aligned}&2^rJSQ^r_{m-1}JSQ^r_n+(4^r+8^r)JSQ^r_{m-2}JSQ^r_{n-1}\\&\quad=J(r,m+n)+2(iJ(r,m+n+1)+jJ(r,m+n+2)\\&\qquad+kJ(r,m+n+3))-J(r,m+n+2)+J(r,m+n+4)\\&\qquad+J(r,m+n+6)\\&\quad=-J(r,m+n)-J(r,m+n+2)+J(r,m+n+4)+J(r,m+n+6)\\&\qquad+2(J(r,m+n)+iJ(r,m+n+1)+jJ(r,m+n+2)\\&\qquad+kJ(r,m+n+3))\\&\quad=2JSQ^r_{m+n}-(J(r,m+n)+J(r,m+n+2)\\&\qquad-J(r,m+n+4)-J(r,m+n+6)).\end{aligned}$$

Hence we get the result. □

Now we will give the generating function for the split r-Jacobsthal quaternion sequence. Similarly as the Jacobsthal sequence, r-Jacobsthal sequence, this sequence can be considered as the coefficients of the power series expansion of the corresponding generating function. We recall the result for the r-Jacobsthal sequence.

Theorem 17 ([2]). *The generating function of the sequence of r-Jacobsthal numbers has the following form:*

$$f(t)=\frac{1+(1+2^r)t}{1-2^rt-(2^r+4^r)t^2}.$$

Theorem 18. *The generating function for the split r-Jacobsthal quaternion sequence $\{JSQ^r_n\}$ has the following form:*

$$g(t)=\frac{JSQ^r_0+(JSQ^r_1-2^rJSQ^r_0)t}{1-2^rt-(2^r+4^r)t^2}.$$

Proof. Let

$$g(t)=JSQ^r_0+JSQ^r_1t+JSQ^r_2t^2+\cdots+JSQ^r_nt^n+\cdots$$

be the generating function of the split r-Jacobsthal quaternion sequence. Then

$$\begin{aligned}2^rtg(t)&=2^rJSQ^r_0t+2^rJSQ^r_1t^2+2^rJSQ^r_2t^3+\cdots\\&\quad+2^rJSQ^r_{n-1}t^n+\cdots,\end{aligned}$$

$$
\begin{aligned}
(2^r+4^r)t^2 g(t) &= (2^r+4^r)JSQ_0^r t^2 + (2^r+4^r)JSQ_1^r t^3 \\
&\quad + (2^r+4^r)JSQ_2^r t^4 + \cdots + (2^r+4^r)JSQ_{n-2}^r t^n + \cdots .
\end{aligned}
$$

By Proposition 6 we get

$$
\begin{aligned}
& g(t) - 2^r t g(t) - (2^r+4^r)t^2 g(t) \\
&\qquad = JSQ_0^r + (JSQ_1^r - 2^r JSQ_0^r)t + (JSQ_2^r - 2^r JSQ_1^r \\
&\qquad\quad - (2^r+4^r)JSQ_0^r)t^2 + \cdots \\
&\qquad = JSQ_0^r + (JSQ_1^r - 2^r JSQ_0^r)t.
\end{aligned}
$$

Thus

$$
g(t) = \frac{JSQ_0^r + (JSQ_1^r - 2^r JSQ_0^r)t}{1 - 2^r t - (2^r+4^r)t^2}.
$$

Using equalities (11), we obtain

$$
\begin{aligned}
JSQ_0^r &= 1 + i(2^{r+1}+1) + j(3\cdot 4^r + 2^{r+1}) \\
&\quad + k(5\cdot 8^r + 5\cdot 4^r + 2^r), \\
JSQ_1^r - 2^r JSQ_0^r &= 2^r + 1 + i(4^r + 2^r) + j(2\cdot 8^r + 3\cdot 4^r + 2^r) \\
&\quad + k(3\cdot 16^r + 5\cdot 8^r + 2\cdot 4^r).
\end{aligned}
$$

□

4. Conclusion. In this study, a one-parameter generalization of the split Jacobsthal quaternions was introduced. Some results including the Binet formula, generating function, a summation formula for these quaternions were given. Moreover, some identities, such as Catalan, Cassini, d'Ocagne and convolution identities, involving the split r-Jacobsthal quaternions were obtained. The presented results are generalization of the known results for the split Jacobsthal quaternions.

5. Acknowledgements. The authors would like to thank the reviewer for helpful valuable suggestions which resulted in improvements to this paper.

References

[1] Akyiğit, M., Kösal, H. H., Tosun, M., *Split Fibonacci quaternions*, Adv. Appl. Clifford Algebr. **23** (2013), 535–545.

[2] Bród, D., *On a new Jacobsthal-type sequence*, Ars Combin., in press.

[3] Cockle, J., *On systems of algebra involving more than one imaginary and on equations of the fifth degree*, Phil. Mag. **35** (3) (1849), 434–435.

[4] Dasdemir, A., *The representation, generalized Binet formula and sums of the generalized Jacobsthal p-sequence*, Hittite Journal of Science and Engineering **3** (2) (2016), 99–104.

[5] Falcon, S., *On the k-Jacobsthal numbers*, American Review of Mathematics and Statistics **2** (1) (2014), 67–77.

[6] Horadam, A. F., *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart. **3** (3) (1965), 161–176.

[7] Horadam, A. F., *Complex Fibonacci numbers and Fibonacci quaternions*, Amer. Math. Monthly **70** (1963), 289–291.

[8] Kilic, N., *On split k-Jacobsthal and k-Jacobsthal–Lucas quaternions*, Ars Combin. **142** (2019), 129–139.
[9] Polatli, E., Kizilates, C., Kesim, S., *On split k-Fibonacci and k-Lucas quaternions*, Adv. Appl. Clifford Algebr. **26** (2016), 353–362.
[10] Tokeşer, Ü., Ünal, Z., Bilgici, G., *Split Pell and Pell–Lucas quaternions*, Adv. Appl. Clifford Algebr. **27** (2017), 1881–1893.
[11] Yağmur, T., *Split Jacobsthal and Jacobsthal–Lucas quaternions*, Commun. Math. Appl. **10** (3) (2019), 429–438.

Dorota Bród
Rzeszów University of Technology
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: dorotab@prz.edu.pl

Received May 14, 2019