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Typically Real Functions Convex in the Direction of the Real Axis

Funkcje typowo rzeczywiste wypukłe w kierunku osi rzeczywistej

Abstract. We consider the class of holomorphic functions univalent on the unit disk that are convex in the direction of the real axis and that have real coefficients. It appears that this class is more complicated than the known class of univalent functions that are convex in the direction of the i-axis and that have real coefficients. For instance, the convexity direction of functions from the first class does not preserve for smaller disks contrary to functions from the second class.

1. Introduction. A plane set D is said to be convex in the direction of a line l (resp. of a vector $e^{i\gamma}$) if for every line l' parallel to l (resp. to $e^{i\gamma}$) the set $D \cap l'$ is either empty or a connected set. Let $H(\Delta)$ denote the class of all complex functions that are holomorphic on the disk $\Delta = \{z : |z| < 1\}$. We say that $f \in H(\Delta)$ is convex in the direction of l if f maps Δ univalently onto a domain convex in the direction of l.

Any function which is convex in one direction can be rotated so that it is convex in the direction of the real or imaginary axis. Let

$$CV(e^{i\gamma}) = \{f \in H(\Delta) : f(0) = f'(0) - 1 = 0, \ f \text{ is convex in the direction of } e^{i\gamma}\}. \tag{1}$$

A representation formula for the set (1) has been found by Royster and Ziegler [9], see also [2], v.I, pp. 193–206. In fact, they used some earlier results of Hengartner and Schober [5] to extend a formula of Robertson [8].

Theorem 1 (Royster, Ziegler). *A function $f \in CV(i)$ if and only if $f \in H(\Delta)$, $f(0) = f'(0) - 1 = 0$ and there are real numbers $\alpha, \beta \in [0, \pi]$ such that*

$$\operatorname{Im}[(e^{i\alpha} - 2z\cos\beta + z^2 e^{-i\alpha})f'(z)] \geq 0 \quad \text{for all } z \in \Delta. \tag{2}$$

Furthermore, if $f \in CV(i)$, then there are $\alpha, \beta \in [0, \pi]$ and sequences (z_n), (ζ_n) in Δ with $z_n \to e^{i(\alpha-\beta)}$, $\zeta_n \to e^{i(\alpha+\beta)}$ as $n \to \infty$ such that (2) *holds and* $\operatorname{Re} f(z_n) \to \sup(\operatorname{Re} f)(\Delta)$, $\operatorname{Re} f(\zeta_n) \to \inf(\operatorname{Re} f)(\Delta)$ *as $n \to \infty$.*

By (2) it follows that $CV(i)$ is the union of a two-parameter family of compact convex sets being affinely homeomorphic to the known class

$$P = \{f \in H(\Delta) : f(0) = 1, \;\; \mathrm{Re}\, f > 0 \text{ on } \Delta\}. \tag{3}$$

Hengartner and Schober [6] observed first that if $f \in CV(i)$ and $0 < r < 1$, this does not imply that the function $z \mapsto f(rz)/r$ is also in $CV(i)$. A concrete example of such a function was given by Goodman and Saff [3], see also [2], vol.1, p. 196. Namely, the univalent function

$$F(z) = [(1-\eta^3 z)^2/(1-\eta z)^2 - 1]/[2(\eta - \eta^3)] \quad \text{with } |\eta| = 1, \; \eta^2 \neq 1,$$

maps Δ onto the complex plane minus a vertical slit, so it belongs to $CV(i)$. However, all the functions $z \mapsto F(rz)/r$ with $\sqrt{2} - 1 < r < 1$ are not in $CV(i)$.

For a class with real coefficients the choice of a convexity direction influences the complexity of the subclass which is distinguished by this direction. Indeed, let us consider the known set

$$TR = \{f \in H(\Delta) : f(0) = f'(0) - 1, \;\; \mathrm{Im}\, f(z)\, \mathrm{Im}\, z \geq 0 \quad \text{for } z \in \Delta\} \tag{4}$$

of all normalized typically real functions and let

$$CVR(e^{i\gamma}) = \{f \in CV(e^{i\gamma}) : f \text{ is real on } (-1,1)\}.$$

The Rogosinski result states that

$$TR = \{f : (1-z^2)f/z \in P \;\text{ and } f \text{ is real on } (-1,1)\}, \tag{5}$$

see [1, 2, 4, 10], and $CVR(i)$ consists of all normalized univalent functions with Steiner symmetric ranges, cp. [5].

The both classes (4) and $CVR(i)$ are joined by the well known

Theorem 2 (Robertson, see [5] and [2], vol.1, p.206). *$f \in CVR(i)$ if and only if $zf' \in TR$ and $f(0) = 0$.*

Hence it follows that for each $f \in CVR(i)$ the property that $f(\Delta)$ is convex in the direction of the i-axis is always preserved for smaller disks. However, for univalent functions with real coefficients the condition to be convex in the direction of the real axis makes our considerations more involved. For instance, the functions

$$\Delta \ni z \mapsto G_\lambda(z) = (\lambda/2)\log[(1+z)/(1-z)] + (1-\lambda)z/(1+z)^2 \tag{6}$$

with $0 \leq \lambda \leq 1$ will be shown to be in $CVR(1)$. But if λ is close to 1, $\lambda < 1$, then the real direction of convexity of G_λ is not preserved for disks close to Δ, see Theorem 3.

In the paper we shall show that the class

$$\Gamma = CVR(1) \tag{7}$$

is the union of a one-parameter family of its compact convex subclasses, each subclass being affinely homeomorphic to (4), and hence having a simple form of its extreme points, see Theorems 4, 5. Let us add that (7) is compact but not convex, so the coefficients regions for (7) are not convex, see Theorem 6.

The Koebe domains for the most often considered classes have been determined, see [2], v. II. Let $A \subset H(\Delta)$. The Koebe domain for A is the collection $K(A)$ of all points w such that w is in $f(\Delta)$ for every $f \in A$, i.e.

$$K(A) = \bigcap_{f \in A} f(\Delta).$$

In most cases $K(A)$ is really a domain what justifies the name "domain". For instance, Reade and Złotkiewicz [7] and later Goodman and Saff [3] proved that

$$K(CV(i)) = \{r\, e^{it} : 0 \le r < 1/[4\sin(3\pi/4 - |t|/2)],\quad -\pi \le t \le \pi\}.$$

In 1977 Goodman found, see [2], vol.2, p.117, that

$$K(TR) = \{r\, e^{it} : 0 \le r < \pi \sin|t|/[4|t|(\pi - |t|)],\ -\pi \le t \le \pi\}.$$

Thus we have $K(\Gamma) \supset i\ K(CV(i)) \cup K(TR)$. In Theorem 7 of our paper we describe the Koebe domain for the class (7).

2. Main results.

Theorem 3. *The functions* (6) *are in* Γ *for all* $0 \le \lambda \le 1$. *If* $(2+\sqrt{3})/4 < \lambda < 1$, *then there exists* r_λ *close to* 1, $r_\lambda < 1$, *such that the functions* $z \mapsto G_\lambda(rz)/r$, $r_\lambda < r < 1$, *do not belong to* Γ.

Proof. Since $\mathrm{Re}\,[(1+z)^2 G'_\lambda(z)] = \mathrm{Re}\,[\lambda(1+z)/(1-z) + (1-\lambda)(1-z)/(1+z)] > 0$ for $z \in \Delta$ and all $0 \le \lambda \le 1$, the function $z \mapsto -iG_\lambda(iz)$ belongs to $CV(i)$, see Theorem 1 with $\alpha = \pi/2$, $\beta = 0$. But G_λ is real on $(-1,1)$ so that $G_\lambda \in \Gamma$ for all $0 \le \lambda \le 1$.

Let now $z = r\,e^{it}$, $0 < r < 1$, $t \in \mathbf{R}$. Observe first that $\partial G_\lambda(z)/\partial t = izG'_\lambda(z)$ and $\partial \operatorname{Im} G_\lambda(z)/\partial t = r(1-r^2)Q(r,t,\lambda)/|(1-z)^2(1+z)^6|$, where

$$Q(r,t,\lambda) \stackrel{df}{=} |1-z|^2[3r + (1+r^2)\cos t - r\cos 2t] + 4\lambda r[2r\cos t + (1+r^2)\cos 2t]. \tag{8}$$

Fix arbitrarily $(2+\sqrt{3})/4 < \lambda < 1$. Then

$$Q(1,t,\lambda) = 4(1+\cos t)[2(1-\lambda) + (4\lambda - 3)\cos t + \cos^2 t]$$

and

$$Q(1,0,\lambda) = Q(1,2\pi,\lambda) = 16\lambda > 0, \tag{9}$$

$$\lim_{t\to\pi} Q(1,t,\lambda)/(1+\cos t) = 24(1-\lambda) > 0, \tag{10}$$

$$Q(1,t_\lambda,\lambda) = Q(1,2\pi - t_\lambda,\lambda) = -(5-4\lambda)(16\lambda^2 - 16\lambda + 1)/2 < 0, \tag{11}$$

where $t_\lambda = \arccos[(3-4\lambda)/2] \in (\arccos[(1-\sqrt{3})/2], 2\pi/3) \subset (7\pi/12, 2\pi/3)$. Moreover, from (8) we get

(12) $$Q(r,\pi,\lambda) = (1-r)^2[4\lambda r - (1+r)^2] < -(1-r)^4 \text{ for } 0 < r < 1.$$

By (10) there is $0 < \delta_\lambda < \pi/6$, e.g. $\delta_\lambda = \arccos(4\lambda - 3)$, for that $Q(1, \pi \pm \delta_\lambda, \lambda) > 0$, so by (9), (11) we have $d_\lambda \overset{df}{=} \min\{Q(1,0,\lambda),\ -Q(1,t_\lambda,\lambda),\ Q(1,\pi-\delta_\lambda,\lambda)\} > 0$. Since the functions $r \mapsto [Q(1,t,\lambda) - Q(r,t,\lambda)]/(1-r)$ are polynomials of the third degree with continuous coefficients, there exists a constant M such that $|Q(r,t,\lambda) - Q(1,t,\lambda)| \le M(1-r)$ for all $0 \le r \le 1$, $0 \le t \le 2\pi$, $0 \le \lambda \le 1$. If now $0 < M(1-r_\lambda) \le d_\lambda$ and $r_\lambda < r < 1$, then

$$Q(r,0,\lambda) = Q(r,2\pi,\lambda) > 0 > Q(r,t_\lambda,\lambda) = Q(r,2\pi - t_\lambda,\lambda),$$
$$Q(r,\pi-\delta_\lambda,\lambda) = Q(r,\pi+\delta_\lambda,\lambda) > 0 > Q(r,\pi,\lambda).$$

This means that for $(2+\sqrt{3})/4 < \lambda < 1$, $r_\lambda < r < 1$ the trigonometric polynomial $t \mapsto Q(r,t,\lambda)$ of the third degree has exactly six changes of sign on the interval $(0, 2\pi)$. Thus, the domains bounded by the level curves $\{G_\lambda(z) : |z| = r\}$ with $(2+\sqrt{3})/4 < \lambda < 1$, $r_\lambda < r < 1$ are not convex in the direction of the real axis.

Theorem 4 (representation of Γ). *$f \in \Gamma$ if and only if $f \in H(\Delta)$, $f(0) = f'(0) - 1 = 0$, f is real on $(-1,1)$, and there exists $\beta \in [0,\pi]$ such that*

(13) $$\operatorname{Re}\{(1 - 2z\cos\beta + z^2)f'(z)\} > 0 \text{ for all } z \in \Delta.$$

Proof. Since $CV(1) = \{z \mapsto -ig(iz) : g \in CV(i)\}$, Theorem 1 implies that $f \in CV(1)$ if and only if $f \in H(\Delta)$, $f(0) = f'(0) - 1 = 0$, and there are $\tau \in [-\pi/2, \pi/2]$ and $\beta \in [0,\pi]$ such that

(14) $$\operatorname{Re}\{e^{i\tau}(1 - 2ze^{-i\tau}\cos\beta + z^2e^{-2i\tau})f'(z)\} \ge 0 \text{ on } \Delta.$$

For $\tau = 0$ the condition (14) becomes (13) so the case "if" is proved.

Let now $f \in \Gamma$. Then for some $\tau \in [-\pi/2, \pi/2]$ and $\beta \in [0,\pi]$ we have (14). Moreover, $f'(z) \equiv \overline{f'(\bar{z})}$, whence

(15) $$\operatorname{Re}\{e^{-i\tau}(1 - 2ze^{i\tau}\cos\beta + z^2e^{2i\tau})f'(z)\} \ge 0 \text{ on } \Delta.$$

Adding (14) and (15) we get

(16) $$\operatorname{Re}\{(\cos\tau - 2z\cos\beta + z^2\cos\tau)f'(z)\} \ge 0 \text{ on } \Delta.$$

For $z = 0$ we have $\cos\tau \ge 0$. If $\cos\tau = 0$, then $\cos\beta = 0$ and $f(z) \equiv (1/2)\log[(1+z)/(1-z)]$, so (13) holds when $\beta = 0$ or $\pi/2$ or else π. Suppose that $\cos\tau > 0$. Since each member of Γ is univalent and has real coefficients, we have the inequality $f'(r) > 0$ for all $-1 < r < 1$. Hence, according to (16),

$$\cos\tau - 2r\cos\beta + r^2\cos\tau \ge 0 \text{ for all } -1 < r < 1.$$

Passing to the limit as $r \to \pm 1$ we get that $|\cos\beta| \le \cos\tau$, and (13) follows from (16) with $\arccos(\cos\beta/\cos\tau)$ instead of β.

Further on let $\mathbf{P}_{[-1,1]}$ denote the set of all probability measures on the interval $[-1,1]$ and let

$$(17)\qquad k(z,t) = z/(1-2tz+z^2)\,,\quad z\in\Delta,\ -1\le t\le 1.$$

It is well known that the set $\{k(\cdot,t) : -1 \le t \le 1\}$ is identical with $E(TR)$, the set of all extreme points of the class TR, and that TR is identical with $\overline{\text{conv}}\, E(TR)$, the closed convex hull of $E(TR)$, see [4, 10].

The full particulars on the set Γ are contained in the

Theorem 5. *Let $c,t\in[-1,1]$ and let*

$$(18)\qquad f_{c,t}(z) = \begin{cases} (2t-2c)^{-1}\log[k(z,t)/k(z,c)] & ,\ t\ne c,\\ \lim\limits_{r\to c} f_{c,r}(z) = k(z,c) & ,\ t=c,\end{cases}$$

where k is defined in (17). *Then*

$$(19)\qquad \Gamma = \bigcup_{-1\le c\le 1}\Gamma_c\,,$$

where

$$(20)\qquad \Gamma_c = \left\{\int_{-1}^{1} f_{c,t}\,d\mu(t) : \mu\in\mathbf{P}_{[-1,1]}\right\}.$$

Moreover,

$$(21)\qquad E\Gamma_c = \{f_{c,t} : -1\le t\le 1\}.$$

Proof. Let $\Phi_f(z) \equiv z^2 f'(z)/[(1-z^2)k(z,c)]$. By Theorem 4 and the property (5) we get (19) with $\Gamma_c = \{f\in H(\Delta) : f(0)=0,\ \Phi_f\in TR\}$. Thus (20) follows from the Robertson formula for the class TR, see [1, 2, 4, 10]. Since $f\mapsto\Phi_f$ is an affine homeomorphism between Γ_c and TR, we have $\Phi(E(\Gamma_c)) = E(TR)$, i.e. (21) holds.

Corollary. $E(\Gamma) = \{k(\cdot,t) : -1\le t\le 1\}$.

Proof. We know that $E(TR) = \{k(\cdot,t) : -1\le t\le 1\} = \{f_{c,c} : -1\le c\le 1\}\subset\Gamma$ and that $\Gamma\subset TR$. Hence $E(TR)\subset E(\Gamma)\subset\{f_{c,t} : -1\le c,t\le 1\}$ by (19), (21). Suppose that $c,t\in[-1,1]$, $c\ne t$. Then

$$f_{c,t} = (1-\lambda)f_{c,x} + \lambda f_{x,t}\notin E(\Gamma)$$

for all x lying in the open interval with end points c and t, where $\lambda = (t-x)/(t-c)\in(0,1)$.

3. Applications.

Theorem 6. *Let $a_j(f) = f^{(j)}(0)/j!$ and let $A_{2,3}(f) = (a_2(f), a_3(f))$. The coefficient region $A_{2,3}(\Gamma)$ is identical with the set*

$$\{(x,y) : -2 \leq x \leq 2,\ x^2 - 1 \leq y \leq (4|x|+1)/3\}.$$

Proof. Let $-1 \leq d \leq 1$. By Theorem 5 we have

$$A_{2,3}(\Gamma_d) = \overline{\text{conv}}\, A_{2,3}(E(\Gamma_d)) = \overline{\text{conv}}\, \{(t+d, -1+4(t^2+td+d^2)/3) : -1 \leq t \leq 1\}$$
$$= \{(x,y) : d-1 \leq x \leq d+1,\ -1+4(x^2-dx+d^2)/3 \leq y \leq (4dx+1)/3\}.$$

Thus

$$A_{2,3}(\Gamma_d) \subset \{(x,y) : -2 \leq x \leq 2,\ x^2-1 \leq y \leq (4|x|+1)/3\}$$
$$= \bigcup_{-1 \leq c \leq 1} \{(x,y) \in A_{2,3}(\Gamma_c) : x = 2c\} \cup A_{2,3}(\Gamma_{-1}) \cup A_{2,3}(\Gamma_1) \subset$$
$$\subset A_{2,3}(\Gamma) = \bigcup_{-1 \leq c \leq 1} A_{2,3}(\Gamma_c).$$

Theorem 7.

$$K(\Gamma) = \{(x,y) : |x| < |y/\pi| \log(-1+\pi/(4|y|)) + 1/4,\ |y| < \pi\lambda_0/4\},$$

where $\lambda_0 = 0.782\ldots$ is the only positive solution of the equation $\lambda \log(1/\lambda - 1) + 1 = 0$. Hence $K(\Gamma)$ is a domain symmetric with respect to the coordinate axes and whose upper and lower halves are convex.

Proof. Observe first that the set $K(\Gamma)$ is convex in the direction of the real axis and symmetric with respect to the coordinate axes (because of conjugation and the real rotation in Γ). Denote the set on the right-hand side by A and let $H_\lambda(z) \equiv -G_\lambda(-z)$, where G_λ is defined in (6) with $0 \leq \lambda \leq 1$. For all $0 < r < 1$, $0 < t < \pi$ we have

$$\text{Re}\, G_\lambda(e^{it}) = [\lambda \log \cot^2(t/2) + (1-\lambda)/\cos^2(t/2)]/4 \geq$$
$$\geq [\lambda \log(1/\lambda - 1) + 1]/4 = \text{Re}\, G_\lambda(1 - 2\lambda + 2i\sqrt{\lambda(1-\lambda)})$$

and $\text{Im}\, G_\lambda(e^{it}) = \lambda\pi/4$. Hence $G_\lambda(\Delta)$ is the plane slit along two horizontal halflines $\{w_\lambda + t : t \geq 0\}$, $\{\overline{w}_\lambda + t : t \geq 0\}$, where $w_\lambda = [\lambda \log(1/\lambda - 1) + 1 + i\lambda\pi]/4$. Thus, by Theorem 3,

$$K(\Gamma) \subset \bigcap_{0 \leq \lambda \leq 1} G_\lambda(\Delta) \cap H_\lambda(\Delta) = A.$$

Since Γ is compact, it follows that to each point a of $\partial K(\Gamma)$, the boundary of $K(\Gamma)$, there corresponds at least one $f_a \in \Gamma$ such that $a \notin f_a(\Delta)$. By symmetry of the set $K(\Gamma)$ we may assume that $\text{Re}\, a \geq 0$. Then f_a is subordinate to G_λ in Δ, where $\lambda = 4\ \text{Im}\, a/\pi$, which means that $f_a = G_\lambda$ so that $\partial K(\Gamma) = \partial A$. For the boundary of $K(\Gamma)$ lying in the first quadrant we have the equation $(0,\lambda_0) \ni \lambda \mapsto w(\lambda) = [1+\lambda\log(1/\lambda-1)+i\pi\lambda]/4$, from which $d\arg w'(\lambda)/d\lambda = |w'(\lambda)|^{-2}\,\mathrm{Im}\,[w''(\lambda)\overline{w'(\lambda)}] = \pi/[16|w'(\lambda)|^2\lambda(1-\lambda)^2] > 0$ and $w'(0^+) = +\infty + i\pi/4$, $w'(1^-) = -\infty + i\pi/4$. The proof is complete.

Remark. (added in proof). After this paper was accepted for publication, we learned that the Koebe domains for the subclasses Γ_{-1}, Γ_1 were found by J. Krzyż and M.O. Reade [Koebe domains for certain classes of analytic functions, Journal D'Analyse Math. 18 (1967), 185–195]. Let us add that $K(\Gamma) = K(\Gamma_{-1}) \cap K(\Gamma_1)$.

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STRESZCZENIE

Rozważamy klasę funkcji holomorficznych i jednolistnych na kole jednostkowym, które są wypukłe w kierunku osi rzeczywistej i które mają rzeczywiste współczynniki. Okazuje się, że ta klasa jest bardziej skomplikowana niż znana klasa funkcji jednolistnych wypukłych w kierunku osi urojonej, których współczynniki są rzeczywiste. Na przykład, kierunek wypukłości funkcji z pierwszej klasy nie zawsze zachowuje się dla mniejszych kół w przeciwieństwie do funkcji z drugiej klasy.