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On Linear Combinations of Convex Functions and Certain Other Holomorphic Mappings

Kombinacje liniowe funkcji wypukłych i innych odwzorowań holomorficznych

Линейные комбинации от выпуклых однолистных функций и некоторых других функций

0. Introduction. For $\rho > 0$ we shall denote the open disk $\{z \in \mathbb{C} : |z| < \rho\}$ by D_ρ and its closure by $\bar{D}_\rho$. The class of all normalized functions $f(z) := z + \sum_{\nu=2}^{\infty} a_\nu z^\nu$ which are convex univalent in D_1 will be denoted by K.

In [3, problem 6.11] it was asked if for $f, g \in K$ and $0 < \lambda < 1$, the combination $\lambda f + (1-\lambda)g$ is starlike univalent in D_1. The question was answered in the negative by MacGregor [4] who pointed out that the functions $f_0(z) := z/(1-ze^{i\pi/4})$, $g_0(z) := z/(1-ze^{-i\pi/4})$ belong to K but $(1/2)f_0(1/\sqrt{2}) + (1/2)g_0(1/\sqrt{2}) = 0$ so that $(1/2)f_0 + (1/2)g_0$ fails to be (locally univalent in $D_{1/\sqrt{2}}$. On the other hand, he noted that for $f_\nu \in K$, $\lambda_\nu \geq 0$ $(\nu = 1,2,\dots,n)$ the function $\sum_{\nu=1}^{n} \lambda_\nu f_\nu$ is always univalent in $D_{1/\sqrt{2}}$. In fact, if $f_\nu \in K$ then [1]

$$(1) \qquad |\operatorname{Arg} f'_\nu(z)| < 2 \arcsin|z| \quad \text{for } z \in D_1$$

which implies that $|\operatorname{Arg} f'_\nu(z)| < \pi/2$ for $|z| < 1/\sqrt{2}$. Hence for $z \in D_{1/\sqrt{2}}$ the real part of $f'_\nu(z)$ is positive and so is the real part of $\sum_{\nu=1}^{n} \lambda_\nu f'_\nu(z)$ if $\lambda_\nu \geq 0$ $(\nu = 1,2,\dots,n)$.

Whereas MacGregor's result tells the truth it does not tell the whole truth. It is intuitively clear that the radius of univalence of $\lambda f + (1-\lambda)g$ must be a continuous function of λ and so cannot suddenly drop from 1 to

$1/\sqrt{2}$ as soon as λ differs from 0 and 1. For given λ in [0,1] let $\Lambda(=\Lambda(\lambda))$ denote the radius of the largest disk centred at the origin in which every function of the family $\{\lambda f + (1-\lambda)g : f \in K,\ g \in K\}$ is univalent. The purpose of this paper is to discuss how Λ depends on λ. Due to obvious symmetry $\Lambda(\lambda) = \Lambda(1-\lambda)$ and so we only need to consider values of λ in [1/2,1].

1. <u>The determination of</u> Λ

According to Dieudonné's criterion [2, p. 310] for univalence, the function $h_\lambda(z) := \lambda f(z) + (1-\lambda)g(z)$ is univalent in D_Λ if and only if

(i) $h_\lambda'(z) \neq 0$,

(ii) $$\frac{h_\lambda(ze^{i\theta}) - h_\lambda(ze^{-i\theta})}{ze^{i\theta} - ze^{-i\theta}} \neq 0 \quad (0 < \theta \leq \pi/2)$$

for $|z| < \Lambda$. Hence $\lambda f + (1-\lambda)g$ is univalent in D_Λ if and only if

(i$'$) $\lambda f'(z) + (1-\lambda)g'(z) \neq 0$,

(ii$'$) $$\lambda \frac{f(ze^{i\theta}) - f(ze^{-i\theta})}{ze^{i\theta} - ze^{-i\theta}} + (1-\lambda)\frac{g(ze^{i\theta}) - g(ze^{-i\theta})}{ze^{i\theta} - ze^{-i\theta}} \neq 0 \quad (0 < \theta \leq \pi/2)$$

for $|z| < \Lambda$. Now let us denote by $G_{\rho,0}$ the set of all possible values of $f'(z)$ as f varies in K and z varies in D_ρ, i.e.

$$G_{\rho,0} := \{f'(z) : f \in K,\ z \in D_\rho\}.$$

Further, for each θ in $(0,\pi/2]$, let

$$G_{\rho,\theta} := \left\{\frac{f(ze^{i\theta}) - f(ze^{-i\theta})}{ze^{i\theta} - ze^{-i\theta}} : f \in K,\ z \in D_\rho\right\}.$$

Finally, for each θ in $[0,\pi/2]$ and $c \in \mathbb{C}$, let $cG_{\rho,\theta}$ denote the set $\{cw : w \in G_{\rho,\theta}\}$.

It is clear that (i$'$), (ii$'$) hold for $|z| < \Lambda$ if and only if for each θ in $[0,\pi/2]$ the sets $\lambda G_{\rho,\theta}$ and $-(1-\lambda)G_{\rho,\theta}$ remain disjoint for $\rho \leq \Lambda$. However, the following lemma implies that $G_{\rho,\theta} \subseteq G_{\rho,0}$ for $0 < \rho < 1$ and $0 < \theta \leq \pi/2$.

Consequently, $\lambda G_{\rho,\theta}$ and $-(1-\lambda)G_{\rho,\theta}$ are disjoint for all θ in $[0,\pi/2]$ if and only if they are disjoint for $\theta = 0$.

LEMMA 1. <u>If</u> $f(z) := z + \sum_{\nu=2}^{\infty} a_\nu z^\nu$ <u>is convex univalent in</u> D_1, <u>then so is</u>

$$F_\theta(z) := \int_0^z \frac{f(\zeta e^{i\theta}) - f(\zeta e^{-i\theta})}{\zeta e^{i\theta} - \zeta e^{-i\theta}}\, d\zeta, \quad 0 < \theta \leq \pi/2.$$

This is a simple consequence of the theorem of Ruscheweyh and Sheil-Small [9, Theorem (2.1)] confirming the Pólya-Schoenberg conjecture and the fact that $w_\theta(z) := \sum_{\nu=1}^{\infty} \frac{1}{\nu} \frac{\sin \nu\theta}{\sin\theta} z^\nu$ belongs to K (the function $W_\theta(z) := \frac{z}{1+2z\cos\theta+z^2} = \sum_{\nu=1}^{\infty} \frac{\sin\nu\theta}{\sin\theta} z^\nu$ being starlike).

It is known that if $f \in K$ then (see for example [8, p. 381] and apply Schwarz's lemma)

$$\left| \frac{1}{\sqrt{f'(z)}} - 1 \right| \leq |z| \quad \text{for } z \in D_1. \tag{2}$$

Consequently,

$$G_{\rho,0} := \left\{ w^2 : w \in \mathbb{C},\ \left| w - \frac{1}{1-\rho^2} \right| < \frac{\rho}{1-\rho^2} \right\}.$$

Using this information it can be shown that the regions $\lambda G_{\rho,0}$ and $-(1-\lambda)G_{\rho,0}$ remain disjoint for $0 < \rho \leq 1/(\sqrt{\lambda} + \sqrt{1-\lambda})$. The details will be presented elsewhere. In fact, we are able to prove the following more general result.

THEOREM 1. <u>Given</u> $0 \leq \alpha < \pi$ <u>let</u> λ_1, λ_2 <u>be complex numbers with</u> $|\operatorname{Arg} \lambda_\nu| \leq \alpha/2$ <u>for</u> $\nu = 1, 2$ <u>and</u> $|\lambda_1| + |\lambda_2| = 1$. <u>If we set</u> $\lambda := \max(|\lambda_1|, |\lambda_2|)$ <u>then for</u> $f_1, f_2 \in K$ <u>the linear combination</u> $\lambda_1 f_1 + \lambda_2 f_2$ <u>is univalent in</u> D_τ <u>where</u> $\tau := \frac{\sqrt{1 - 2\sqrt{\lambda(1-\lambda)}\sin\frac{\alpha}{2}}}{\sqrt{\lambda} + \sqrt{1-\lambda}}$. <u>The result is sharp for each</u> α <u>and each</u> λ.

In order to see that Theorem 1 is indeed sharp let $\lambda \in [1/2, 1)$ and $\alpha \in [0, \pi)$ be given. Then the functions

$$f_1(z) := \frac{z}{1 - ze^{-i\gamma}}, \quad f_2(z) := \frac{z}{1 - ze^{-i\delta}}$$

where

$$\tau \cos\gamma = \frac{\sqrt{1-\lambda} - \sqrt{\lambda}\sin\frac{\alpha}{2}}{\sqrt{\lambda} + \sqrt{1-\lambda}}, \quad \tau \sin\gamma = \frac{-\sqrt{\lambda}\cos\frac{\alpha}{2}}{\sqrt{\lambda} + \sqrt{1-\lambda}},$$

$$\tau \cos\delta = \frac{\sqrt{\lambda} - \sqrt{1-\lambda}\sin\frac{\alpha}{2}}{\sqrt{\lambda} + \sqrt{1-\lambda}}, \quad \tau \sin\delta = \frac{\sqrt{1-\lambda}\cos\frac{\alpha}{2}}{\sqrt{\lambda} + \sqrt{1-\lambda}}$$

belong to K and

$$\lambda e^{i\alpha/2} f_1'(\tau) + (1-\lambda)e^{-i\alpha/2} f_2'(\tau) = 0.$$

Consideration of complex coefficients λ_1, λ_2 was inspired by [11].

2. Convex linear combinations of convex mappings

The reasoning of Section 1 can be easily adapted to deal with linear combinations of several convex mappings. Here is what we obtain:

THEOREM 2. *Let* $\lambda_\nu \geq 0$ *for* $\nu = 1, \ldots, n$ *with* $\sum_{\nu=1}^{n} \lambda_\nu = 1$ *and suppose that* $\lambda := \max_{1\leq\nu\leq n} \lambda_\nu \geq 1/2$. *Further, let* $\Lambda := 1/(\sqrt{\lambda} + \sqrt{1-\lambda})$ *and* $\Omega := (\lambda+1)/\left\{2\lambda + \sqrt{2(1-\lambda)}\right\}$. *If* $f_1, \ldots, f_n$ *belong to* K, *then* $\sum_{\nu=1}^{n} \lambda_\nu f_\nu$ *is univalent in* D_Λ *for* $1/2 \leq \lambda \leq (1/2)\left\{(2-\sqrt{2})(1+\sqrt{1+4\sqrt{2}})\right\}^{1/2}$, $(1/2)^{1/4} \leq \lambda \leq 1$ *and in* D_Ω *for* $(1/2)\left\{(2-\sqrt{2})(1+\sqrt{1+4\sqrt{2}})\right\}^{1/2} \leq \lambda \leq (1/2)^{1/4}$. *The result is sharp for each* λ.

If $\lambda := \max_{1\leq\nu\leq n} \lambda_\nu < 1/2$ then in the case of even n the best that can be said is that $\sum_{\nu=1}^{n} \lambda_\nu f_\nu$ is univalent in $\overline{D}_{1/\sqrt{2}}$. The same remark applies if n is *odd* provided $\lambda \geq 1/(n-1)$.

3. Functions starlike of order 1/2

The function $f(z) := z + \sum_{\nu=2}^{\infty} a_\nu z^\nu$, holomorphic in D_1, is said to be starlike of order 1/2 if $\operatorname{Re}\{zf'(z)/f(z)\} > 1/2$ for all $z \in D_1$. The usual notation for the set of all such functions is $S^*_{1/2}$. According to a result of Strohhäcker [10] $K \subset S_{1/2}$. We observe that Theorems 1 and 2 remain true in the wider class $S^*_{1/2}$. In fact, the conclusions of those theorems depend entirely on two properties of the class K, namely (i) if f belongs to the class then so does the function F_θ (introduced in Lemma 1) for $0 < \theta \leq \pi/2$, (ii) for each function f belonging to the class, f' is subordinate to I' where $I(z) := z/(1-z)$. Using a result of Ruscheweyh and Sheil-Small [[9, Theorem (3.1)] about the Hadamard product of functions starlike of order 1/2 we can prove that in Lemma 1 the words "convex univalent" may be replaced by "starlike of order 1/2". This means that the class $S^*_{1/2}$ has property (i). That it also has property (ii) is a result of Pfaltzgraff [7].

4. Linear combinations of polynomials

Our approach to the above mentioned problem is of considerably wider scope. It can not only be applied to the study of the linear combinations of functions belonging to various other families of univalent functions but can also be used to obtain the following result about polynomials.

THEOREM 3. *Given* $0 \leq \beta < \pi/2$ *let* λ_1, λ_2 *be complex numbers with* $|\operatorname{Arg} \lambda_\nu| \leq \beta$ *for* $\nu = 1, 2$ *and* $|\lambda_1| + |\lambda_2| = 1$. *If* $\lambda := \max(|\lambda_1|, |\lambda_2|)$ *and*

$$f_\mu(z) := 1 + \sum_{\nu=1}^{n} a_{\mu,\nu} z^\nu, \quad (\mu = 1, 2)$$

are polynomials of degree at most n *not vanishing in* D_1, *then* $\lambda_1 f_1(z) + \lambda_2 f_2(z)$ *does not vanish in* D_σ, *where*

$$\sigma := \frac{\sqrt{\lambda^{2/n} + (1-\lambda)^{2/n} - 2\lambda^{1/n}(1-\lambda)^{1/n}\cos((\pi-2\beta)/n)}}{\lambda^{1/n} + (1-\lambda)^{1/n}}.$$

The result is sharp for each β *and each* λ.

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STRESZCZENIE

Jak wiadomo, kombinacja liniowa $\lambda f + (1 - \lambda)g$ funkcji wypukłych, $0 < \lambda < 1$, nie musi być funkcją wypukłą. Jednakże przy danym λ kombinacja liniowa ma określony promień jednolistności $\Lambda(\lambda)$. W pracy wyznaczono dokładną wartość Λ. Rozwiązano też problem analogiczny dla zespolonych λ_1, λ_2, takich że $|\lambda_1|+|\lambda_2| = 1$.

РЕЗЮМЕ

Как известно, линейная комбинация $\lambda f + (1-\lambda) g$ от выпуклых функций не всегда является выпуклой. Однако для данного λ эта комбинация имеет определенный радиус однолистности $\Lambda(\lambda)$. В этой работе получено точное значение Λ. Также решена аналогическая проблема для комплексных λ_1, λ_2 исполняющих $|\lambda_1|+|\lambda_2|=1$.