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On Semi-dual Analytic Functions

O funkcjach analitycznych semi-dualnych

Аналитические семи-дуальные функции

1. Introduction. Let A denote the set of all analytic functions in $D := \{z: |z| < 1\}$ and A_0 the subset of all functions $f \in A$ with $f(0) = 1$. Using the Hadamard product $f * g$ for $f, g \in A$ we define the dual set V^* of some $V \subset A_0$ as

$$V^* := \{f \in A_0 : (f * g)(z) \neq 0 \text{ for all } z \in D \text{ and } g \in V\} .$$

We call a subset $W \subset A_0$ dual if it is the dual set of some (other) subset of A_0. Let now $V \subset A_0$. The smallest dual set which contains V is called the dual hull of V and denoted by du (V). This concept was introduced in [4] and by means of the "duality principle" it could be shown that many known results on certain classes of functions in D have an immediate interpretation in terms of dual sets. As well, a number of new results were obtained using duality theory. Dual sets have many properties similar to convex sets but in some respects they seem to be better adjusted to characteristics of analytic functions. On the other hand, an intrinsec definition of dual sets is still missing, let alone a theory of the Krein-Milman type. The present paper is one step towards a better understanding of dual sets. We are dealing with the study of "small" duals sets, namely those which are of the form

$$\{f(xz) : |x| \leq 1\} \quad (1.1)$$

where $f \in A_0$. It is known that the sets (1.1) are dual for some f and not dual for others. A general rule is not yet known, however, some partial results are available which seem to indicate a close connection of this problem with entire functions and exceptional values. We call $f \in A_0$ semi-dual if (1.1) is a dual set. We first list some known results.

Let T be a subset of $N_0 := N \cup \{0\}$ with $0 \in T$ and

$$A_T := \left\{ f \in A_0 : f(z) = \sum_{k \in T} a_k z^k , \quad a_k \neq 0 \text{ for } k \in T \right\}. \quad (1.2)$$

In particular,

$$e_T(z) := \sum_{k \in T} z^k \in A_T . \quad (1.3)$$

<u>Theorem A (cf. [3]).</u> $f \in A_T$ <u>is semi-dual iff</u> e_T <u>is semi-dual. Therefore, the question of the semi-duality of</u> f <u>depends only on the gap-structure of the power-series expansion of</u> f <u>at</u> $z = 0$. <u>We note that</u> e_T <u>is trivially semi-dual for</u> $T = \{0, 1\}$.

Theorem B (cf. [3]). Assume that $T \neq \{0, 1\}$ but $\sum_{k \in T} 1/k < \infty$. Then e_T is not semi-dual.

Theorem C (cf. [3]). Let $n \in N$ and

$$(1.4) \qquad T_n := \{kn : k \in N_0\} .$$

Then e_T is semi-dual.

In [3] V. Kasten and St. Ruscheweyh mentioned some working hypotheses and tenative conjectures, all of them based on the following general assumption: if $T \subset T'$ and e_T is semi-dual, then $e_{T'}$ is semi-dual. Although this sounds plausible it turns out to be wrong. Our results below show that, for instance, $T = T_2$ and $T' = T_2 \cup \{1\}$ provide a counterexample. Also, with the same T', we easily see that $\{e_T\}^* \cap \tilde{A}_T$, where $\tilde{A}_{T'} = \bigcup_{T \subset T'} A_T$ is not compact in any disc $|z| \leq \rho$, $0 < \rho < 1$, which disproves another of the conjectures in [3]. The main idea, however, namely the existence of a relation between semi-duality of e_T and the existence of entire functions $f \in \tilde{A}_T$ with $f \neq 0$ in C seems to remain intact, but in a slightly modified form. We formulate this as a problem:

Problem. Let $T \neq \{0, 1\}$. Is it true that e_T is semi-dual if and only if for every $k_0 \in T$ there exist T_0 with $k_0 \in T_0 \subset T$ such that A_{T_0} contains an entire function non-vanishing in C ?

The following three theorems support the statement above:

Theorem 1. Let S, T be such that e_S, e_T are semi-dual. If either $1 \in S \cap T$ or $\{a,b\} \subset S \cap T$, where the greatest common divisor of a and b is 1, then $e_{T'}$ is semi-dual for $T' = S \cup T$.

Theorem 2. Let T be a (finite or infinite) union of sets T_n as defined in (1.4). Then e_T is semi-dual.

Theorem 3. Let $T \neq \{0, 1\}$. If e_T is semi-dual, then T contains infinitely many even numbers. T also contains either no or infinitely many odd numbers.

We observe that the functions e_T of Theorem 2 are so far the only known semi-dual functions. Theorem 3 shows that there are many non-semi-dual functions which do not satisfy the condition of Theorem B. We belive that Theorem 3 can be extended to the statement given in the following conjecture:

Conjecture 1. If e_T is semi-dual, then for $b \in T$ and $d \in N$ the set $T \cap \{b + kd : k \in N\}$ is infinite.

Note that Theorem 3 is the case $d = 2$ of the above conjecture, which would be a consequence of the following more general conjecture on non-vanishing functions in D.

Conjecture 2. Let $F := \{f \in A_0 : f \neq 0 \text{ in } D\}$. Let

$m, n \in N$ and $c_1, \ldots, c_m \in C$. Then there exists a constant $M(c_1, \ldots, c_m; n)$ such that for every polynomial P of degree $\leqslant n$ with representation

$$P(z) = \sum_{k=1}^{m} c_k f_k(z) \quad , \quad f_k \in F \qquad (k=1,\ldots,m)$$

the inequality

$$|P(z)| \leqslant M(c_1, \ldots, c_m; n) \qquad (z \in D)$$

holds.

In this paper we prove Conjecture 2 for $m = 2$. This is the content of

Theorem 4. Let $a, b \in C$ and $n \in N$. Then there exists a constant $M(a,b;n)$ such that $|P(z)| \leqslant M(a,b;n)$ for every polynomial P of degree $\leqslant n$ which has a representation $P(z) = af(z) + bg(z)$, $z \in D$ and $f, g \in F$.

More closely related to Conjecture 1 is a recent result of Hayman:

Theorem D. (cf. [2]). Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be non-constant and entire and assume there exist $b \in N_0$, $d \in N$ such that $a_b \neq 0$ and $a_{b+kd} = 0$, $k \in N$. Then f assumes every complex number as a value in C.

Theorem D together with an affirmative solution of the Problem would also establish Conjecture 1.

2. Proofs. Theorem 1 and 2 are applications of Theorem C and some elementary number theory. We recall from [3] that e_T is semi-dual if and only if

$$V_T = V_T^{**} := (V_T^*)^*$$

where $V_T = \{e_T(xz) : |x| \leq 1\}$. Note that

$$V_T^* = \{f \in A_0 : e_T * f \in F\} .$$

Proof of Theorem 1. Let $T' = S \cup T$ and $g \in V_{T'}^{**}$. For $h \in V_S^*$ we have $h * e_S \in F$ and thus $h * e_S \in V_{T'}^*$. Therefore

$$(h * e_S) * g = h * (e_S * g) \in F$$

and we conclude

$$g * e_S \in V_S^{**} .$$

Similarly, $g * e_T \in V_T^{**}$, and since e_S, e_T are semi-dual we have

$$(g * e_S)(z) = e_S(xz) , \quad (g * e_T)(z) = e_T(yz)$$

where $x, y \in \overline{D}$. For the coefficients of z^a, z^b in the power series expansion of g we thus obtain

$$x^a = y^a \quad , \quad x^b = y^b$$

and, from the assumption on the greatest common divisor of a and b , it follows $x = y$. Hence $(g * e_T)(z) = e_T(xz)$ and by a standard application of the "duality principle" [4] we finally deduce $g(z) = e_T(xz)$ which completes the proof.

Proof of Theorem 2. Let $T = \{n_j\}$, $0 = n_0 < n_1 < n_2 < \dots$, and $T(j) = T_{n_j}$, $j \in N$. For $g \in V_T^{**}$ we obtain as in the proof of Theorem 1

$$(g * e_{T(j)})(z) = e_{T(j)}(x_j z) \quad , \quad x_j \in \bar{D} \quad .$$

Also, by the same reasoning as above, g has an expansion of the form

$$g(z) = \sum_{n \in T} a_n z^n$$

and thus

$$a_{kn_j} = x_j^{kn_j} \quad , \quad j \in N \quad , \quad k \in N \, .$$

In particular,

$$(2.1) \qquad a_{n_r n_s} = x_r^{n_r n_s} = x_s^{n_r n_s} \quad , \quad r \, , s \in N$$

and hence

$$|x_r| = |x_s| = t \quad , \quad \text{say.}$$

If $t = 0$ we have $g \equiv 1$ and hence $g \in V_T$, the desired conclusion. Let $t \neq 0$. By a suitable rotation y_1, $|y_1| = 1$, we have

$$g(y_1 z) = 1 + t^{n_1} z^{n_1} + \text{higher terms.}$$

We now proceed by induction, applying further rotations again and again. Assume we have found a rotation y_s, $|y_s| = 1$, such that $g(y_s z)$ has the form

$$(2.2) \qquad 1 + t^{n_1} z^{n_1} + \ldots + t^{n_s} z^{n_s} + \text{higer terms} \quad .$$

In fact, we may assume that (2.2) is g since pure rotations (i.e., with modulus 1) do not interfere with our goal. Hence

$$(2.3) \qquad x_j = t \quad , \quad j = 1, \ldots, s \quad ,$$

and we set

$$a_{n_{s+1}} = e^{2\pi i \varphi} t^{n_{s+1}} \quad , \quad 0 \leqslant \varphi \leqslant 1 \; .$$

If $(b_1, \ldots, b_n)$, $[b_1, \ldots, b_n]$ denote the greatest common divisor and the least common multiple of b_j, $j=1, \ldots, n$, we get from (2.1)

$$(2.4) \qquad e^{2\pi i \varphi [n_j, n_{s+1}]/n_{s+1}} = 1 \quad , \quad j = 1, \ldots, s \quad .$$

If $\varphi = 0$ we have $a_{n_{s+1}} = t^{n_{s+1}}$ and no rotation is required to arrive at the next induction step. If $\varphi \neq 0$, we see from (2.4) that

$$(2.5) \qquad [n_j , n_{s+1}] \varphi / n_{s+1} \in N , \quad j = 1 , \ldots , s .$$

We write $\varphi / n_{s+1} = p/q$ with $p , q \in N$ and $(p , q) = 1$. Thus

$$q \mid [n_j , n_{s+1}] , \quad j = 1 , \ldots , s$$

and this implies

$$q \mid [(n_1 , \ldots , n_s) , n_{s+1}]$$

and

$$mq = [(n_1 , \ldots , n_s) , n_{s+1}] = \frac{(n_1 , \ldots , n_s) n_{s+1}}{((n_1 , \ldots , n_s) , n_{s+1})}$$

for a certain $m \in N$. This shows that φ has a representation

$$\varphi = mp((n_1 , \ldots , n_s) , n_{s+1}) / (n_1 , \ldots , n_s) .$$

There exist numbers $k , l \in Z$ such that

$$((n_1 , \ldots , n_s) , n_{s+1}) = k(n_1 , \ldots , n_s) + ln_{s+1}$$

and hence

$$\varphi = mpk + \frac{mpl \, n_{s+1}}{(n_1 , \ldots , n_s)} .$$

We choose

$$y = e^{-2\pi i m p l/(n_1, \dots, n_s)}$$

so that

$$y^{n_j} = 1 \quad , \quad j = 1, \dots, s$$

$$y^{n_{s+1}} = e^{-2\pi i \varphi} \quad .$$

Thus

$$g(yz) = \sum_{j=0}^{s+1} t^{n_j} z^{n_j} + \text{higher terms} \quad .$$

Induction and a standard convergence argument shows now the existence of an $\tilde{y}$ with $|\tilde{y}| = 1$ such that

$$g(\tilde{y}z) = e_T(tz)$$

which is our assertion.

In order to prove Theorem 3 we first establish a somewhat stronger result, namely Theorem 4.

Proof of Theorem 4. The bounds $M(0, a; n) = M(a, 0; n) = |a| \cdot 2^n$ are well-known. Thus assume $a, b \neq 0$. Then for the function $h(z) = P(z)/f(z)$ we have:

$$(2.5) \quad \begin{cases} h \text{ is analytic in } D, \\ h \text{ has at most } n \text{ zeros in } D, \\ h \neq a, \quad z \in D, \\ h(0) = a + b \quad . \end{cases}$$

The set of all functions satisfying (2.5) is normal (cf. [1], p. 70, Th. 2), in fact, locally uniformly bounded since $h(0)$ is fixed:

$$|h(z)| \leqslant M_1(a\ ,\ b\ ;\ n)\quad ,\quad |z| \leqslant 1/2$$

or, in the same disk

$$|f(z)| \geqslant M_2(a\ ,\ b\ ;\ n)\cdot|P(z)|\quad ,\quad M_2 = 1/M_1\quad .$$

Let

$$\rho_k := \frac{n+1+k}{4n+4}\quad ,\quad k = 0\ ,\ 1\ ,\dots,\ n\ ,$$

so that $\rho_k \in [1/4\ ,\ 1/2]$. Then for $|z| \leqslant 1/4$ we have by the minimum principle for $|f(z)|$

$$|f(z)| \geqslant M_2\cdot\max_k \min_{|z|=\rho_k} |P(z)|\quad .$$

Assume first that

$$m := \max_k \min_{|z|=\rho} |P(z)| \geqslant 1\quad . \tag{2.6}$$

Then f belongs to the locally uniformly bounded family of analytic functions in $|z| < 1/4$ with $f(0) = 1$, $f(z) \neq 0$, and $f(z) \neq M/2$, say. Thus there exists a constant M_3 such that

$$|f(z)| \leqslant M_3\quad ,\quad |z| \leqslant 1/8\quad ,$$

and similar consideration gives a constant M_4 such that

$$|g(z)| \leqslant M_4 \quad , \quad |z| \leqslant 1/8 \quad .$$

Hence

$$|P(z)| \leqslant |a| \cdot M_3 + |b| \cdot M_4 \quad , \quad |z| \leqslant 1/8$$

and we find a constant M_5 such that

(2.7) $$|P(z)| \leqslant M_5 \quad \text{for } z \in D \quad .$$

However, if $m \leqslant 1$ (see (2.6)) we have

(2.8) $$|P(z)| \leqslant 6^n (n+1)^{n+1} \quad , \quad z \in D \quad .$$

To see this we let z_k be the points on $|z| = \rho_k$ where $|P(z)|$ attains its minimum, so that $|P(z_k)| \leqslant 1$. Let $Q(z) = \prod_{k=0}^{n} (z - z_k)$. By Lagrange's interpolation formula we have

$$P(z) = \sum_{k=0}^{n} \frac{P(z_k)}{Q'(z_k)} \cdot \frac{Q(z)}{z - z_k} \quad .$$

Now

$$|Q'(z_k)| = \prod_{\substack{j=0 \\ j \neq k}}^{n} |z_j - z_k| \geqslant \left(\frac{1}{4n+4}\right)^n$$

$$\left|\frac{Q(z)}{z - z_k}\right| = \sum_{\substack{j=0 \\ j \neq k}}^{n} |z - z_j| \leqslant (3/2)^n \quad , \quad |z| = 1$$

which give (2.8). Finally, (2.7) and (2.8) combine to yield the assertion of Theorem 4.

Proof of Theorem 3. Assume first that T has only finitely many even numbers. Then for $f \in V_T^* \cap A_T$ we have

$$f(z) + f(-z) = P(z)$$

where P is a polynomial of a degree which is less or equal to the largest even number n in T. By Theorem 4, applied to f and $g := f(-z)$, $a = 1$, $b = 1$ we see that

$$|P(z)| \leqslant M(1, 1; n) \quad , \quad z \in D .$$

This implies

$$|a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leqslant \frac{1}{2} M(1, 1; n)$$

for $f \in V_T^*$ and therefore

$$g(z) := 1 + \frac{2}{M(1, 1; n)} z^n \in V_T^{**}$$

which contradicts the semi-duality of e_T.

The proof of the "odd" case runs similary.

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STRESZCZENIE

Niech A będzie rodziną wszystkich funkcji analitycznych w kole jednostkowym $\mathbb{D}$ i niech $A_0 = \{f \in A : f(0) = 1\}$. Jeśli $V \subset A_0$ to zbiór dualny V^* zbioru V jest określony jako zbiór tych $f \in A_0$, że splot Hadamarda $f * g$ nie zeruje się w $\mathbb{D}$ dla dowolnej funkcji $g \in V$. Podzbiór $W \subset A_0$ nazywamy dualnym, jeśli istnieje $U \subset A_0$ taki, że $W = U^*$. Funkcję $f \in A_0$ nazywam semi-dualną, jeśli zbiór $\{f(xz) : |x| \le 1\}$ jest dualny. Niech $e_T(z) = 1 + \sum_{k \in T} z^k$, gdzie $T \subset \mathbb{N}$. W pracy badane są związki pomiędzy strukturą zbioru T i semidualnością funkcji e_T.

РЕЗЮМЕ

Пусть A семейство всех функций аналитических в единичном круге $\mathbb{D}$, $A_0 = \{f \in A : f(0) = 1\}$. Если $V \subset A_0$, тогда дуальное множество V^* это множество всех $f \in A_0$ таких, что свертка Гадамара $f * g$ не равна нулю в $\mathbb{D}$ для любой функции $g \in V$. Подмножество $W \subset A_0$ называется дуально, если найдется $U \subset A_0$ такое, что $W = U^*$. Функция $f \in A_0$ называется семидуальной, когда множество $\{f(xz) : |x| \leq 1\}$ дуально. Пусть $e_T(z) = 1 + \sum_{k \in T} z$, где $T \subset N$.

В данной работе занимаемся отношением между структурой множества T и семидульностей функции e_T.