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On the Ahlfors Class N in an Annulus

O klasie N Ahlforsa dla pierścienia

Об Альфорсовом классе N для кольца

Introduction. To show that the theory of quasiconformal mappings is not an ad hoc generalization of the theory of conformal mappings, but is, on the contrary intimately tied to the classical theory Ahlfors [1] has investigated the class N of complex-valued L^{∞} functions ν in the unit disk for which the antilinear part of the Fréchet differential of normalized quasiconformal mappings vanishes, where the mappings are generated by complex dilatation of the form $t\nu$, t being a real parameter. He gave there a theory of this class and showed that the complex structure of Teichmüller space of closed Riemann surfaces of genus $g > 1$ carries a natural complex analytic structure which can be derived from the corresponding structure of L^{∞} by means of generalized Riemann mapping theorem.

The theory of this class N has been used by Reich and Strebel [4] in connection with one of the most important extremal problems in the unit disk concerning the functions with given boundary values.

Very deep investigation of the class N has been given by

Reich in [3], where he considered also the class N in an annulus with "inward extension".

This class N has also been investigated by Ławrynowicz [2] and Zając [5] and used as a tool to obtain a parametric representation of Teichmüller quasiconformal mappings of an annulus [6]. The results presented here have an expository character. We present also some new results due to the first author concerning equivalence condition for functions of this class.

1. The class N_r. Let μ be a complex-valued measurable function in an annulus $\Delta_r = \{z : r \leq |z| \leq 1\}$, $0 \leq r < 1$, which satisfies the condition

$$\|\mu\|_\infty = \inf_{E} \sup_{z \in \Delta_r \setminus E} |\mu(z)| < 1$$

where the infimum is taken over all sets of the plane measure zero. It is well-known that there exists exactly one number R, $0 \leq R < 1$, and one Q-quasiconformal mapping f of the annulus Δ_r onto Δ_R which satisfies the Beltrami equation

(1) $$f_{\bar{z}} = \mu f_z \quad \text{with} \quad f(1) = 1,$$

where $Q = (1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$.

Suppose now that $\mu = t\nu$, where $\|\nu\|_\infty < \infty$, and $0 \leq t < 1/\|\nu\|_\infty$. Denote explicitly the dependence of f on ν: $f(z,t) = f[\nu](z,t)$, $r \leq |z| \leq 1$. Let

(2) $$\dot{f}[\nu](z) = \lim_{t \to 0} \frac{1}{t}\left\{f[\nu](z,t) - z\right\},$$

which is a Fréchet differential of $f[\nu]$. This expression is well defined and depends linearly on ν (cf. [1]). From $f[\nu]_{\bar{z}} = t\nu f[\nu]_z$ it turns out that $\dot{f}[\nu]$, regarded as a function of z, has partial derivatives almost everywhere, and in particular it satisfies the differential equation

$$(3) \qquad \dot{f}[\nu]_{\bar z} = \nu \quad .$$

It is well-known that (3) is satisfied only if

$$(4) \qquad \dot{f}[\nu](\zeta) = -\frac{1}{\pi}\iint_{\Delta_r} \frac{\nu(z)}{z-\zeta}\,dxdy + F(\zeta)$$

with holomorphic F .

Thus we have (cf. [2])

$$(5) \qquad \dot{f}[\nu](\zeta) = \frac{\zeta}{2\pi}\iint_{\Delta_r} \sum_{k=-\infty}^{+\infty}\left[\frac{\nu(z)}{z^2}\left(\frac{\zeta+r^{2k}z}{\zeta-r^{2k}z} - \frac{1+r^{2k}z}{1-r^{2k}z}\right) - \right.$$

$$\left. - \frac{\overline{\nu(z)}}{z^2}\left(\frac{1+r^{2k}\zeta\bar z}{1-r^{2k}\zeta\bar z} - \frac{1+r^{2k}\bar z}{1-r^{2k}\bar z}\right)\right] dxdy \quad .$$

We see that $\dot f$ is a linear continuous operator which maps every $\nu \in L^\infty(\Delta_r)$ on a function $\dot f[\nu]$. As it is shown in [2] the relations $|f[\nu](z,t)| = 1$ for $|z| = 1$, and $|f[\nu](z,t)| = R[\nu](t)$ $|z| = r$ yield

$$(6) \qquad \operatorname{Re}\left\{\bar z\dot f[\nu](z)\right\} = \begin{cases} 0 & \text{for } |z| = 1 , \\ r\varrho & \text{for } |z| = r , \end{cases}$$

where $\varrho = \lim\limits_{t\to 0} \frac{1}{t}\left\{R[\nu](t) - r\right\}$. In analogy to the above we can verify that

$$(7) \qquad \operatorname{Re}\left\{\bar z\dot f[i\nu](z)\right\} = \begin{cases} 0 & \text{for } |z| = 1 , \\ r\varrho^* & \text{for } |z| = r , \end{cases}$$

where $\varrho^* = \lim\limits_{t\to 0} \frac{1}{t}\left\{R[i\nu](t) - r\right\}$. For more details see [2].

We recall [2] that

$$(8) \qquad \varrho = \frac{r}{2\pi}\iint_{\Delta_r}\left[\frac{\nu(z)}{z^2} + \frac{\overline{\nu(z)}}{\bar z^2}\right] dxdy$$

by which

$$(9)\qquad \varrho^{*} = \frac{ir}{2\pi} \iint_{\Delta_r} \left[\frac{\vartheta(z)}{z^2} - \frac{\overline{\vartheta(z)}}{\bar{z}^2}\right] dxdy \ .$$

Following Ahlfors [1] let us decompose the Frechet differential $\dot{f}[\vartheta]$ defined by (2) as follows

$$(10)\qquad \dot{f}[\vartheta] = \frac{1}{2}\left\{\dot{f}[\vartheta] + i\dot{f}[i\vartheta]\right\} + \frac{1}{2}\left\{\dot{f}[\vartheta] - i\dot{f}[i\vartheta]\right\} ,$$

where the first part is antilinear and the second one is linear with respect to the complex multipliers. By the definition of $\dot{f}[\vartheta]$ we can see that $\left\{\dot{f}[\vartheta] + i\dot{f}[i\vartheta]\right\}_{\bar{z}} = 0$ i.e.

$$(11)\qquad \Phi[\vartheta] = \dot{f}[\vartheta] + i\dot{f}[i\vartheta]$$

is always a holomorphic function. The antilinearity is expressed by $\Phi[i\vartheta] = -i\Phi[\vartheta]$.

We denote by N_r the subspace of $L^{\infty}(\Delta_r)$ which is formed by all ϑ with $\Phi[\vartheta] = 0$. It is a complex linear subspace of $L^{\infty}(\Delta_r)$. Now we can state

<u>Theorem 1. An element</u> ϑ <u>of</u> $L^{\infty}(\Delta_r)$ <u>belongs to</u> N_r <u>if and only if one of the following assumptions hold:</u>

$$(12)\qquad \dot{f}[\vartheta](\zeta) = \begin{cases} 0 & \underline{\text{for}}\quad |\zeta| = 1 , \\ \dfrac{\zeta}{\pi} \displaystyle\iint_{\Delta_r} \frac{\vartheta(z)}{z^2}\, dxdy & \underline{\text{for}}\quad |\zeta| = r , \end{cases}$$

$$(13)\qquad \dot{f}[\vartheta](\zeta) = \frac{\zeta}{\pi} \iint_{\Delta_r} \sum_{k=-\infty}^{+\infty} \frac{\vartheta(z)}{z^2} \left[\frac{\zeta + r^{2k}z}{\zeta - r^{2k}z} - \frac{1+r^{2k}z}{1-r^{2k}z}\right] dxdy \ ,$$

$$(14)\qquad \iint_{\Delta_r} \vartheta(z)g(z)dxdy = \frac{1}{2\pi} \iint_{\Delta_r} \frac{\vartheta(z)}{z^2}\, dxdy \int_{|z|=r} zg(z)dz$$

<u>for all</u> g <u>holomorphic in</u> $\operatorname{int}\Delta_r$ <u>with</u> $\iint_{\Delta_r} |g(z)|\,dxdy < \infty$.

<u>Proof.</u> The proof of (12) is presented in details in [5] and [2]. The condition (13) is an immediate consequence of (5) and the definition of the class N_r . To get the condition (14) suppose that g is holomorphic in $\operatorname{int}\Delta_r$ with finite L^1 norm in Δ_r . Then by (12) and Green's formulae we have the equality (14). Conversely, if (14) is fulfilled, then we apply it to $g(z) = \frac{1}{\pi}(\zeta - z)^{-1}$, $|\zeta| = r$, and next when $|\zeta| = 1$. Because

$$\int_{|z|=r} zg(z)dz = \int_{|z|=r+\varepsilon} zg(z)dz \quad , \quad \text{where } 0 < \varepsilon < 1-r$$

which follows by an approximation argument applied to classical Green's formulae. By this (14) is valid as soon as

$\iint_{\Delta_r} |g(z)|\,dxdy < \infty$. Then

$$\int_{|z|=r+\varepsilon} \frac{z}{z-\zeta}\,dz = \begin{cases} 0 & \text{for } |\zeta| = 1 , \\ 2\pi i\zeta & \text{for } |\zeta| = r . \end{cases} \tag{15}$$

It shows that the right side of (14) has the same boundary values as it is given by (12). Making use of the integral representation given by the formulae (4) we see that the boundary values of $\dot{f}[\nu]$ are those of an holomorphic function, which by the normalization condition vanishes at $z=1$, so it must be identically zero and we conclude (12).

<u>2. Other properties of the class</u> N_r . Suppose that

$$\nu(\varrho e^{i\theta}) = \sum_{n=-\infty}^{+\infty} \alpha_n(\varrho)e^{in\theta} \quad , \quad r < \varrho < 1 \tag{16}$$

which is the Fourier series of ν . Let now g be as in Theorem 1 and let

$$(17)\qquad g(z) = \sum_{k=-\infty}^{+\infty} a_k z^k = \sum_{k=-\infty}^{+\infty} a_k \varrho^k e^{ik\theta}, \quad z=\varrho e^{i\theta}, \ r<\varrho<1$$

be its Laurent series. Now, by the argument given in the proof of Theorem 1, we may express (14) in terms of the coefficients $\alpha_n(\varrho)$ and a_k, $n,k=0, \pm 1, \pm 2, \dots$. By this we have

$$(18)\qquad \iint_{\Delta_r} \vartheta(z)g(z)dxdy = 2\pi\int_r^1 \left\{\sum_{n=-\infty}^{+\infty} \alpha_n(\varrho)a_{-n}\varrho^{1-n}\right\}d\varrho =$$

$$= 2\pi \sum_{n=-\infty}^{+\infty} a_{-n}\int_r^1 \alpha_n(\varrho)\varrho^{1-n}\,d\varrho .$$

For the right side of (14) we have

$$(19)\qquad \frac{i}{2\pi}\iint_{\Delta_r}\frac{\vartheta(z)}{z^2}dxdy \int_{|z|=r} zg(z)\,dz =$$

$$= -\frac{1}{2\pi}\iint_{\Delta_r}\frac{\vartheta(z)}{z^2}dxdy\int_{|z|=r} z^2 g(z)\frac{dz}{iz} =$$

$$= -2\pi a_{-2}\int_r^1 \frac{\alpha_2(\varrho)}{\varrho}d\varrho .$$

Let

$$A_{n,k} = \int_r^1 \alpha_n(\varrho)\varrho^{k+1}d\varrho ,$$

then by (23) and (24) the equality (14) can be expressed in the form

$$(20)\qquad \sum_{n=-\infty}^{+\infty} a_{-n}A_{n,-n} = -a_{-2}A_{2,-2} .$$

Let $H(D)$ denote the Banach space of all holomorphic functions with finite L^1-norm in a domain D . If $D_1 \subset D_2$, then clearly $H(D_1) \supset H(D_2)$.

In the case of the unit disk it is easy to see that the unit disk can be replaced by an arbitrary simply connected region D. If $D_1 \subset D_2$ and $\nu \in N(D_1)$, then $\tilde{\nu} \in N(D_2)$, where

$$\tilde{\nu}(z) = \begin{cases} \nu(z) , & z \in D_1 \\ 0 , & z \in D_2 \setminus D_1 . \end{cases}$$

Making use of (14) we see that previous implication is also true in the case of a doubly connected domain.

These results have a natural analog in the case $r = 0$, i.e. for mappings in the unit disk with an additional invariant point zero.

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STRESZCZENIE

W pracy 1 Ahlfors wprowadził podklasę N klasy L^{∞} funkcji zespolonych ν w kole jednostkowym, takich, że dla odwzorowania quasikorforemnego generowanego przez dylatację $t\nu$, $t \in \mathbb{R}$, znika identycznie część antyliniowa jego różniczki Frécheta.

Autorzy badają własności funkcji należących do analogicznej klasy funkcji w pierścieniu.

РЕЗЮМЕ

В работе [1] Альфорсем введен класс $N \subset L^{\infty}$ комплексных функций ν в единичном круге, таких что для квазиконформного отображения порожденного комплексной дильатацией $t\nu$, $t \in \mathbb{R}$ антилинейная часть его дифференциала Фреще равна нулю.

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