ANNALES UNIVERSITATIS MARIAE CURIE-SKLODOWSKA

LUBLIN-POLONIA

VOL.XXXIX,2 SECTIO A 1985

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Almost r-paracontact Structures

Struktury prawie r-parakontaktowe

Почти r-параконтактные структуры

In this paper we introduce the notion of the almost r-paracontact structure on a manifold M, which is the generalization of the almost paracontact structures. We define the notion of normality of this structure and give its geometric interpretation. Every almost r-paracontact structure induces, in a natural way, some almost paracontact structures whose normality is closely related to the one of the initial structure. We also give some examples of almost r-paracontact structures. Such structures, in a natural way, appear while lifting of an almost paracontact structure to the tangent bundle.Manifolds and tensor fields, being under consideration throughout the paper are of the class C^∞.

Definition 1. If, on a manifold M, there exist a tensor field ϕ of type (1.1) and r vector fields $\xi_1, \xi_2, \dots, \xi_r$ and r 1-forms $\eta^1, \eta^2, \dots, \eta^r$ such, that:

$$\eta^i(\xi_j) = \delta^i_j, \quad i, j = 1, 2, \dots, r, \tag{1}$$

$$\phi(\xi_i) = 0, \quad i = 1, 2, \dots, r, \tag{2}$$

$$\eta^i \circ \phi = 0, \quad i = 1, 2, \dots, r, \tag{3}$$

$$\phi^2 = \mathrm{Id} - \sum_{i=1}^{r} \eta^i \otimes \xi_i \tag{4}$$

then, the structure $\Sigma = (\phi, \xi_1, \xi_2, \dots, \xi_r, \eta^1, \eta^2, \dots, \eta^r)$ is said to be an almost r-paracontact structure on M.

If, moreover, on M , there exists a positive definite Riemannian metric g such that :

$$\bigwedge_{X\in V(M)} \eta^i(X) = g(X,\xi_i) \quad i=1,\ldots,r, \tag{5}$$

$$\bigwedge_{X,Y\in V(M)} g(\phi X,\phi Y) = g(X,Y) - \sum_{i=1}^{r} \eta^i(X)\eta_i(Y) \tag{6}$$

($V(M)$ denotes the set of all vector fields on M), then $\Sigma = (\phi,\xi_{(i)},\eta^{(i)},g)_{i=1,\ldots,r}$ is called an almost r-paracontact metric structure on M. The metric g is called compatible Riemannian metric.

Lemma 1. [5]. *Let $\xi_1,\xi_2,\ldots,\xi_r$ and $\eta^1,\eta^2,\ldots,\eta^r$ be r vector fields and r 1-forms on a manifold M respectively, such that the condition* (1) *is satisfied. Then there exists a positive definite Riemannian metric g on M satisfying the condition* (5).

Theorem 1. *Let $\Sigma = (\phi,\xi_{(i)},\eta^{(i)})_{i=1,\ldots,r}$ be an almost r-paracontact structure on M. Then M admits a positive definite Riemannian metric G satisfying the conditions* (5) *and* (6).

Proof. According to Lemma 1 we can find a metric g satisfying the condition (5).

Let

$$G(X,Y) = \frac{1}{2}\left\{g(X,Y) + g(\phi X,\phi Y) + \sum_i \eta^i(X)\eta^i(Y)\right\}.$$

Obviously we have $G(X,\xi_i) = \eta^i(X)$ and $G(\xi_i,\xi_j) = \delta_{ij}$. Then:

$$\begin{aligned} G(\phi X,\phi Y) &= \frac{1}{2}\left\{g(\phi X,\phi Y) + g(\phi^2 X,\phi^2 Y) + \sum_i \eta^i(\phi X)\eta^i(\phi Y)\right\} \\ &= \frac{1}{2}\left\{g(\phi X,\phi Y) + g(X,Y) + \sum_{ij} \eta^i(X)\eta^j(Y)g(\xi_i,\xi_j) - \right. \\ &\left. -\eta^i(X)g(\xi_i Y) - \eta^i(Y)g(X,\xi_i)\right\} = \\ &= G(X,Y) - \sum_i \eta^i(X)\eta^i(Y) \end{aligned}$$

or G satisfies the condition (6).

Remark 1. Observe, that the restriction of ϕ to the subspace

$$\{X : \eta^i(X) = 0, i = 1,2,\ldots,r\}$$

of TM satisfies the condition: $G(X,\phi Y) = G(\phi X,Y)$. Hence the eigenvalues of ϕ are real and equaled 0, 1, -1 .

Analogously to the case of paracontactness ([3],[4]), we have the following:

Theorem 2. *On a manifold M there is one-to-one correspondence between almost r-paracontact metric structures on M and the reductions of the structural* *group of the tangent bundle of M to the subgroup* $1 \times \cdots \times 1 \times O(n-p-r) \times O(p)$, *where p is the multiplicity of the eigenvalue* 1 *of the characteristic equation of* ϕ.

Now, we deal with the normality of the almost r-paracontact structure on a manifold M. Let $\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}$ be an almost r-paracontact structure on M. Denoting by $(t^1, \dots, t^r)$ the canonical coordinates on $\mathbf{R}^r$ we can define on $N = M \times \mathbf{R}^r$ the following tensor field:

$$F(Y) = F\left(X + \sum_i f^i \frac{d}{dt^i}\right) = \left(\phi X + \sum_i f^i \xi_i + \sum_i \eta^i(X) \frac{d}{dt^i}\right) \tag{7}$$

for every vector field

$$Y = X + \sum_i f^i \frac{d}{dt^i} \in V(N) \text{ where } X \in V(M) .$$

Remark 2. From now on, we'll be omitting the sign $\sum_i$ and the summation convention will be used.

F is the tensor field of an almost product structure on N because:

$$\begin{aligned} F^2(Y) &= F^2\left(X + f^i \frac{d}{dt^i}\right) = F\left(\phi X + f^i \xi_i + \eta^i(X) \frac{d}{dt^i}\right) = \\ &= \left(\phi\left(\phi X + f^i \xi_i\right) + \eta^i(X)\xi_i + \eta^j\left(\phi X + f^i \xi_i\right) \frac{d}{dt^i}\right) = \\ &= \left(\phi^2 X + f^i \phi(\xi_i) + \eta^i(X)\xi_i + \left((\eta^j \circ \phi)(X) + f^i \eta^j(\xi_i)\right) \frac{d}{dt^i}\right) = \\ &= \left(X - \eta^i(X)\xi_i + \eta^i(X)\xi_i + f^i \frac{d}{dt^i}\right) = \left(X + f^i \frac{d}{dt^i}\right) = Y. \end{aligned}$$

For any $A, B \in V(N)$, the value of the Nijenhuis tensor field of F is:

$$N_F(A,B) = [A,B] + [FA,FB] - F[A,FB] - F[FA,B] . \tag{8}$$

Definition 2. An almost r-paracontact structure $\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}$ on a manifold M is said to be normal if and only if the almost product structure F defined by (7) on $M \times \mathbf{R}^r$ is integrable i.e. $N_F = 0$.

Now, since N_F is the tensor field, then the vanishing of N_F on $M \times \mathbf{R}^r$ is equivalent to the vanishing of N_F on:

(*i*) $A, B \in V(M)$, (*ii*) $A \in V(M), B \in V(\mathbf{R}^r)$, (iii) $\mathrm{A}, \mathrm{B} \in \mathrm{V}(\mathbf{R}^r)$.

(i) Let $X, Y \in V(M)$, then:

$$
\begin{aligned}
N_F(X,Y) &= [X,Y] + [FX,FY] - F[X,FY] - F[FX,Y] = \\
&= [X,Y] + [\phi X + \eta^i(X)\frac{d}{dt^i}, \phi Y + \eta^i(Y)\frac{d}{dt^i}] - \\
&-F[X, \phi Y + \eta^i(Y)\frac{d}{dt^i}] - F[\phi X + \eta^i(X)\frac{d}{dt^i}, Y] = \\
&= [X,Y] + [\phi X, \phi Y] + [\phi X, \eta^i(Y)\frac{d}{dt^i}] + [\eta^i(X)\frac{d}{dt^i}, \phi Y] + \\
&+ \left[\eta^i(X)\frac{d}{dt^i}, \eta^j(Y)\frac{d}{dt^j}\right] - F([X,\phi Y]) - F\left(\left[X, \eta^i(Y)\frac{d}{dt^i}\right]\right) - \\
&-F([\phi X, Y]) - F\left(\left[\eta^i(X)\frac{d}{dt^i}, Y\right]\right) = \\
&= [X,Y] + [\phi X, \phi Y] + \phi(X)\left(\eta^i(Y)\right)\frac{d}{dt^i} - (\phi Y)\left(\eta^i(X)\right)\frac{d}{dt^i} - \\
&-\phi[X,\phi Y] - \eta^i[X,\phi Y]\frac{d}{dt^i} + X(\eta^i(Y))F\left(\frac{d}{dt^i}\right) - \\
&-\phi[\phi X, Y] - \eta^i[\phi X, Y]\frac{d}{dt^i} - Y(\eta^i(X))F\left(\frac{d}{dt^i}\right) = \\
&= [X,Y] + [\phi X, \phi Y] - \phi[X,\phi Y] - \phi[\phi X, Y] - \\
&-\eta^i[X,Y]\xi_i - \{X(\eta^i(Y)) - Y(\eta^i(X)) - \eta^i[X,Y]\}\xi_i + \\
&+ \{(\phi X)(\eta^i(Y) - \eta^i[\phi X, Y] - (\phi Y)(\eta^i(X)) + \eta^i[\phi Y, X]\}\frac{d}{dt^i} .
\end{aligned}
$$

We have:

$$
N_\phi(X,Y) = [X,Y] + [\phi X, \phi Y] - \phi[X,\phi Y] - \phi[\phi X, Y] - \eta^i[X,Y]\xi i . \qquad (9)
$$

This is the Nijenhuis tensor of the almost r-paracontact structure

$$
\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}.
$$

Making use of the following:

$$
2d\eta^i(X,Y) = X(\eta^i(X,Y)) - Y(\eta^i(X)) - \eta^i[X,Y] , \; i = 1,2,\dots,r \qquad (10)
$$

$$
(\alpha_X \eta^i)(Y) = X(\eta^i(Y)) - \eta^i[X,Y] \qquad (11)
$$

$$
(\alpha_X \phi)(Y) = [X,\phi Y] - \phi[X,Y] \qquad (12)
$$

where α_X denotes the Lie derivative with respect to a vector field X, we have:

$$
N_F(X,Y) = N_\phi(X,Y) - 2d\eta^i(X,Y)\xi_i + \{(\alpha_{\phi X}\eta^i)(Y) - (\alpha_{\phi Y}\eta^i)(X)\}\frac{d}{dt^i}.
$$

In the second case i.e. if $X \in V(M)$ and $B = \frac{d}{dt^i}$ we have :

$$N_F\left(X, \frac{d}{dt^i}\right) = \left[FX, F\left(\frac{d}{dt^i}\right)\right] + \left[X, \frac{d}{dt^i}\right] - F\left[X, F\left(\frac{d}{dt^i}\right)\right] - F\left[FX, \frac{d}{dt^i}\right] =$$
$$= \left[\phi X + \eta^j(X)\frac{d}{dt^j}, \xi_i\right] - \phi[X, \xi_i] - \eta^j[X, \xi_i]\frac{d}{dt^j} =$$
$$= [\phi X, \xi_i] + \eta^j(X)\left[\frac{d}{dt^j}, \xi_i\right] - \xi_i(\eta^j(X))\frac{d}{dt^j} - \phi[X, \xi_i] - \eta^j[X, \xi_i]\frac{d}{dt^j} =$$
$$= -([\xi_i, \phi X] - \phi[\xi_i, X]) - (\xi_i(\eta^j(X)) - \eta^j[\xi_i, X])\frac{d}{dt^j} =$$
$$= -(\alpha_{\xi_i}\phi)(X) - (\alpha_{\xi_i}\eta^j)(X)\frac{d}{dt^j} .$$

If $A = \frac{d}{dt^i}$ and $B = \frac{d}{dt^j}$ we have:

$$N_F\left(\frac{d}{dt^i}, \frac{d}{dt^j}\right) = \left[F\frac{d}{dt^i}, F\frac{d}{dt^j}\right] + \left[\frac{d}{dt^i}, \frac{d}{dt^j}\right] - F\left[\frac{d}{dt^i}, F\frac{d}{dt^j}\right] - F\left[F\frac{d}{dt^i}, \frac{d}{dt^j}\right] = [\xi_i, \xi_j]$$

Put :

$$\overset{1}{N}(X,Y) = N_\phi(X,Y) - 2d\eta^i(X,Y)\xi_i , \quad X, Y \in V(M) , \tag{13}$$

$$\overset{2}{N}{}^i(X,Y) = (\alpha_{\phi X}\eta^i)(Y) - (\alpha_{\phi Y}\eta^i)(X) , \quad X, Y \in V(M) , \tag{14}$$

$$\overset{3}{N}_i(X) = -(\alpha_{\xi_i}\phi)(X) , \quad X \in V(M) , \tag{15}$$

$$\overset{4}{N}{}^j_i = -(\alpha_{\xi_i}\eta^j)(X) , \quad X \in V(M) . \tag{16}$$

Now, we can write the values of N_F in all three cases as follows :

$$\text{(i)} \qquad N_F(X,Y) = \overset{1}{N}(X,Y) + \overset{2}{N}{}^i(X,Y)\frac{d}{dt^i} , \quad X, Y \in V(M) , \tag{17}$$

$$\text{(ii)} \qquad N_F\left(X, \frac{d}{dt^i}\right) = \overset{3}{N}_i(X) + \overset{4}{N}{}^j_i(X)\frac{d}{dt^j} , \quad X \in V(M) , \tag{18}$$

$$\text{(iii)} \qquad N_F\left(\frac{d}{dt^i}, \frac{d}{dt^j}\right) = [\xi_i, \xi_j] \qquad i, j = 1, 2, \ldots, r . \tag{19}$$

Hence we have:

Theorem 3. *An almost r-paracontact structure Σ on M is normal if and only if:*

$$\overset{1}{N} = \overset{2}{N}{}^i = \overset{3}{N}_i = \overset{4}{N}{}^j_i = [\xi_i, \xi_j] = 0, \quad i, j = 1, 2, \ldots, r.$$

Now we prove:

Theorem 4. *If* $\overset{1}{N} = 0$, *then*

$$\overset{2}{N}{}^i = \overset{3}{N}_i = \overset{4}{N}{}_i^j = [\xi_i, \xi_j] = 0 \,, \quad i,j = 1,2,\dots,r \,.$$

Proof. Suppose that $\overset{1}{N} = 0$. For ξ_i and ξ_j we have:

$$\overset{1}{N}(\xi_i, \xi_j) = [\xi_i, \xi_j] - \eta^k[\xi_i, \xi_j]\xi_k + \eta^k[\xi_i, \xi_j]\xi_k = [\xi_i, \xi_j]$$

thus, because of the assumption we get:

$$\bigwedge_{i,j=1,2,\dots,r} [\xi_i, \xi_j] = 0. \tag{20}$$

It is easy to verify, that:

$$FN_F(A,B) = -N_F(A,FB)\,, \qquad A,B \in V(N)\,. \tag{21}$$

This relation gives the following identities:

$$FN_F(X,Y) = -N_F(X,FY)\,, \qquad X,Y \in V(M)\,, \tag{22}$$

$$FN_F\left(X, \frac{d}{dt^i}\right) = -N_F(X,\xi_i)\,, \qquad X \in V(M), \tag{23}$$

$$FN_F\left(\frac{d}{dt^i}, X\right) = -N_F\left(\frac{d}{dt^i}, FX\right)\,, \qquad X \in V(M)\,, \tag{24}$$

$$FN_F\left(\frac{d}{dt^i}, \frac{d}{dt^j}\right) = -N_F\left(\frac{d}{dt^i}, \xi_j\right)\,. \tag{25}$$

From (22) we have:

$$\begin{aligned} \phi\overset{1}{N}(X,Y) \;&+\; \overset{2}{N}{}^i(X,Y)\xi_i + \eta^i\left(\overset{1}{N}(X,Y)\right)\frac{d}{dt^i} = \\ &= \; -\overset{1}{N}(X,\phi Y) - \overset{2}{N}{}^i(X,\phi Y)\frac{d}{dt^i} - \eta^i(Y)(\overset{3}{N}_i(X) - \eta^i(Y)\overset{4}{N}{}_i^j(X)\frac{d}{dt^j}. \end{aligned}$$

Hence we get:

$$\phi\overset{1}{N}(X,Y) + \overset{1}{N}(X,\phi Y) + \overset{2}{N}{}^i(X,Y)\xi_i + \eta^i(Y)\overset{3}{N}_i(X) = 0\,. \tag{26}$$

From (23) we have:

$$\begin{aligned} \phi\overset{3}{N}_i(X) \;&+\; \eta^j\left(\overset{3}{N}_i(X)\right)\frac{d}{dt^j} + \overset{4}{N}{}_i^j(X)\xi_j = \\ &= -\overset{1}{N}(X,\xi_i) - \overset{2}{N}{}^j(X,\xi_i)\frac{d}{dt^j}. \end{aligned}$$

Hence:

$$\phi \overset{3}{N}_i(X) + \overset{1}{N}(X,\xi_i) + \overset{4}{N}{}^j_i(X)\xi_j = 0 . \tag{27}$$

From (24) we have:

$$\phi \overset{3}{N}_i(X) \quad + \quad \eta^j \left(\overset{3}{N}_i(X)\right) \frac{d}{dt^j} + \overset{4}{N}{}^j_i(X)\xi_j =$$
$$= -\overset{3}{N}_i(\phi X) - \overset{4}{N}{}^j_i(\phi X)\frac{d}{dt^j} + \eta^j(X)[\xi_i,\xi_j].$$

Hence we get:

$$\phi \overset{3}{N}_i(X) + \overset{3}{N}_i(\phi X) + \overset{4}{N}{}^j_i(X)\xi_j - \eta^j(X)[\xi_i,\xi_j] = 0 . \tag{28}$$

From the identity (25) we have:

$$\phi[\xi_i,\xi_j] + \eta^k [\xi_i,\xi_i] \frac{d}{dt^k} = \overset{3}{N}_i(\xi_j) + \overset{4}{N}{}^k_i (\xi_j) \frac{d}{dt^k} .$$

Hence

$$\phi[\xi_i,\xi_j] = \overset{3}{N}_i(\xi_j) . \tag{29}$$

Acting with η^k on (27) we obtain:

$$\overset{4}{N}{}^j_i(X) = -\eta^j \overset{1}{N}(X,\xi_i)$$

and because of the assumption we get:

$$\overset{4}{N}{}^j_i = 0 . \tag{30}$$

From (27) because of the assumption and (30) we have:

$$\phi \overset{3}{N}_i(X) = 0 .$$

From (28) we have:

$$\overset{3}{N}_i(\phi X) = 0 . \tag{31}$$

From (29) because of (20) we have:

$$\overset{3}{N}_i(\xi_j) = 0 . \tag{32}$$

Since every vector field $X \in V(M)$ is a combination of ϕX and ξ_j so from (31) and (32) we obtain:

$$\overset{3}{N}_i = 0 . \tag{33}$$

Having acted with η^k on (26) we have:

$$\overset{2}{N^i}(X,Y) = \eta^i \overset{1}{N}(X,\phi Y) - \eta^j(Y)\eta^i(\overset{3}{N}_i(X)) .$$

Now, because of the assumption and (33) we obtain:

$$\overset{2}{N^i} = 0 \tag{34}$$

and this completes the proof.

We'll need one more identity being useful in the next part of the paper. From (27) and (28) we have:

$$\overset{1}{N}(X,\xi_i) = \overset{3}{N}_i(\phi X) - \eta^j(X)[\xi_i,\xi_j],$$

Now, if we insert ϕX instead of X into the above, we get:

$$\overset{1}{N}(\phi X,\xi_i) = \overset{3}{N}_i(X) - \eta^j(X)\overset{3}{N}_i(\xi_j)$$

and on account of (29) we have:

$$\overset{1}{N}(\phi X,\xi_i) = \overset{3}{N}_i(X) - \eta^j(X)\phi[\xi_i,\xi_j]. \tag{35}$$

Combining Theorems 3 and 4 we obtain:

Theorem 5. *An almost r-paracontact structure Σ on M is normal if and only if* $\overset{1}{N} = 0$.

For every manifold M with an almost r-paracontact structure, on account of Remark 1, we can define the following distributions:

$$\begin{aligned} D^0 &= \{X;\phi X = 0\}, & \dim D^0 &= r \\ D^+ &= \{X;\phi X = X\}, & \dim D^+ &= p \\ D^- &= \{X;\phi X = -X\}, & \dim D^- &= q, \qquad p+q+r=n . \end{aligned} \tag{36}$$

Analogously as in [2] we have:

Lemma 2.

$$\begin{aligned} &\text{a)} \quad X \in D^+ \oplus D^0 \Leftrightarrow \phi X = X - \eta^i(X)\xi_i , \\ &\text{b)} \quad X \in D^- \oplus D^0 \Leftrightarrow \phi X = -X + \eta^i(X)\xi_i . \end{aligned}$$

Now we deal with the relation between the tensor $\overset{1}{N}$ and the integrability of the above distributions. For $X,Y \in D^+$ we have:

$$\overset{1}{N}(X,Y) = [X,Y] - \phi[X,Y] + [X,Y] - \phi[X,Y]$$

or

$$\phi[X,Y] = [X,Y] - \frac{1}{2}\overset{1}{N}(X,Y) . \tag{37}$$

For $X, Y \in D^-$ we have:

$$\overset{1}{N}(X,Y) = [X,Y] + [X,Y] + \phi[X,Y] + \phi[X,Y]$$

or

$$\phi[X,Y] = -[X,Y] + \frac{1}{2}\overset{1}{N}(X,Y). \tag{38}$$

We'll use the obvious:

Lemma 3. *Let D^+ (resp. D^-) be integrable and each $[\xi_i, \xi_j] = 0$. The distribution $D^+ \oplus D^0$ (resp. $D^- \oplus D^0$) is integrable if and only if for $X \in D^+$ (resp. $X \in D^-$) $[X, \xi_i] \in D^+ \oplus D^0$ (resp. $[X, \xi_i] \in D^- \oplus D^0$).*

For $X \in D^+$ we have:

$$\overset{1}{N}(X, \xi_i) = [X, \xi_i] - \phi[X, \xi_i] .$$

Hence we get:

$$\begin{aligned} \phi[X, \xi_i] &= [X, \xi_i] - \eta^j[X, \xi_i]\xi_j - \overset{1}{N}(X, \xi_i) + \eta^j[X, \xi_i]\xi_j = \\ &= [X, \xi_i] - \eta^j[X, \xi_i]\xi_j - \overset{1}{N}(X, \xi_i) - \eta^j[\xi_i, X]\xi_j . \end{aligned}$$

After having used (35), (11), (16) and the assumption of Lemma 3, we obtain: For $X \in D^+$

$$\phi[X, \xi_i] = [X, \xi_i] - \eta^j[X, \xi_i]\xi_j - \overset{3}{N}_i(X) - \overset{4}{N}{}_i^j(X)\xi_j . \tag{39}$$

Similarly, for $X \in D^-$ we have:

$$\phi[X, \xi_i] = -[X, \xi_i] + \eta^j[X, \xi_i]\xi_j + \overset{3}{N}_i(X) + \overset{4}{N}{}_i^j(X)\xi_j . \tag{40}$$

Now we'll prove the following:

Theorem 6. *In an almost r-paracontact manifold the tensor $\overset{1}{N} = 0$ if and only if each $[\xi_i, \xi_j] = 0$ and $\overset{4}{N}{}_i^j = 0$ and the distributions D^+, D^-, $D^+ \oplus D^0$, $D^- \oplus D^0$ are integrable.*

Proof. Let $\overset{1}{N} = 0$, then from Theorem 4 $[\xi_i, \xi_j] = 0$ $i,j = 1,2\ldots,r$ $\overset{2}{N}{}^i = \overset{3}{N}_i = \overset{4}{N}{}_i^j = 0$, and because of (37), (38), (39), (40), Lemma 2 and Frobenius's theorem, we obtain that the distributions D^+, D^-, $D^+ \oplus D^0$, $D^- \oplus D^0$ are integrable. Now, conversely we can prove that if $\overset{4}{N}{}_i^j = 0$, $[\xi_i, \xi_j] = 0$, $i, j = 1,2\ldots,r$ and the above distributions are all integrable, then the tensor $\overset{1}{N}$ is identically zero.

To this end, since $\overset{1}{N}$ is a tensor field, it suffices to prove the vanishing of $\overset{1}{N}$ in the following four cases:

$$
\begin{array}{ll}
(i) & X,Y \in D^+ \vee X,Y \in D^- , \\
(ii) & X \in D^+ \wedge Y \in D^0 \vee X \in D^- \wedge Y \in D^0 , \\
(iii) & X \in D^+ \wedge Y \in D^- , \\
(iv) & X,Y \in D^0 .
\end{array}
$$

The first case is obvious, because of (37) and (38). In the second case, for every ξ_i and $X \in D^+$ from (35), (39), and $\overset{4}{N}{}_i^j = 0$, $[\xi_i, \xi_j] = 0$, $i,j = 1,2\ldots,r$, we have:

$$0 = \overset{3}{N}_i(X) = \overset{1}{N}(\phi X, \xi_i) = \overset{1}{N}(X, \xi_i).$$

and this means that for any $X \in D^+$ and $Y \in D^0$

$$\overset{1}{N}(X,Y) = 0.$$

Similarly, we have for $X \in D^-$ and $Y \in D^0$

$$\overset{1}{N}(X,Y) = 0.$$

In the case (iii), for $X \in D^+$ and $Y \in D^-$

$$\overset{1}{N}(X,Y) = [X,Y] - [X,Y] + \phi[X,Y] - \phi[X,Y] = 0.$$

In the last case, for $X = \xi_i$ and $Y = \xi_j$ we have:

$$\overset{1}{N}(X,Y) = \overset{1}{N}(\xi_i, \xi_j) = [\xi_i, \xi_j] = 0.$$

Combining Theorems 5 and 6 we obtain the geometric interpretation of normality in the following:

Theorem 7. *An almost r-paracontact structure* Σ *on a manifold* M *is normal if and only if* $\overset{4}{N}{}_j^i = 0$, $[\xi_i, \xi_j] = 0$, $i,j = 1,2,\ldots,r$ *and the distributions* D^+, D^-, $D^+ \oplus D^0$, $D^- \oplus D^0$ *are integrable.*

Remark. Proceeding in the similar way we can give the geometric interpretation of integrability of an almost product structure, namely: Let F be a tensor field of an almost product structure on a manifold. Then there exists a positive definite Riemannian metric g such that $g(FX,FY) = g(X,Y)$ or $g(X,FY) = g(FX,Y)$ and this means that all eigenvalues of F are real and equal to 1 or -1. Defining the distributions $D^{+F} = \{X; FX = X\}$ and $D^{-F} = \{X; FX = -X\}$ we obtain that F is integrable if and only if D^{+F} and D^{-F} are integrable.

Now, we show that on a manifold M with an almost r-paracontact structure Σ there exist almost paracontact and product structures whose normality depends on the normality of the structure Σ.

Let $\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}$ be an almost r-paracontact structure on M. Here we use the following notations: $\xi_r = \xi, \eta^r = \eta$ and so $\Sigma = (\phi, \xi_a, \xi, \eta^a, \eta)_{a=1,\dots,r-1}$. We have:

Theorem 8. *$\Sigma_1 = (\Phi, \xi, \eta)$ where $\Phi = \phi - \eta^a \otimes \xi_a$ is an almost r-paracontact structure on M.*

Now we'll prove the following:

Theorem 9. *If an almost r-paracontact structure Σ on M is normal then the almost paracontact structure $\Sigma_1 = (\phi - \eta^a \otimes \xi_a, \xi, \eta)$ on M is normal.*

Proof. For the structure Σ_1 we have:

$$\begin{aligned}
\overset{1}{N}{}^{\Sigma_1}(X,Y) &= [X,Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] - \\
&\quad -\eta[X,Y]\xi - 2d\eta(X,Y)\xi = \\
&= [X,Y] + [\phi X - \eta^a(X)\xi_a, \phi Y - \eta^\beta(Y)\xi_\beta - \\
&\quad -\Phi[X, \phi Y - \eta^a(Y)\xi_a] - \Phi[\phi X - \eta^a(X)\xi_a, Y] - \\
&\quad -\eta[X,Y]\xi - 2d\eta(X,Y)\xi = \\
&= [X,Y] + [\phi X, \phi Y] - \phi[X, \phi Y] - \phi[\phi X, Y] - \\
&\quad -\eta^a(X)[\xi_a, \phi Y] + (\phi Y)(\eta^a(X))\xi_a - \eta^a(Y)[\phi X, \xi_a] - \\
&\quad -(\phi X)(\eta^a(Y))\xi_a + \eta^a(X)\xi_a(\eta^\beta(Y))\xi_\beta - \eta^\beta(Y)\xi_\beta(\eta^a(X))\xi_a + \\
&\quad +\eta^a[X, \phi Y]\xi_a + \eta^a(Y)\phi[X, \xi_a] - \eta^a(Y)\eta^\beta[X, \xi_a]\xi_\beta - \\
&\quad -X(\eta^a(Y))\xi_a + \eta^a[\phi X, Y]\xi_a + \eta^a(X)\phi[\xi_a, Y] - \\
&\quad -\eta^a(X)\eta^\beta[\xi_a, Y]\xi_\beta + Y(\eta^a(X))\xi_a - \eta[X,Y]\xi - \\
&\quad -2d\eta(X,Y)\xi + \eta^a(X)\eta^\beta(Y)[\xi_a, \xi_\beta] = \\
&= \overset{1}{N}(X,Y) - \overset{2}{N}{}^a(X,Y)\xi_a + \eta^a(X)\overset{3}{N}_a(Y) - \\
&\quad -\eta^a(Y)\overset{3}{N}_a(X) - \eta^a(X)\overset{4}{N}{}^\beta_a(Y)\xi_\beta + \\
&\quad +\eta^a(Y)\overset{4}{N}{}^\beta_a(X)\xi_\beta + \eta^a(X)\eta^\beta(Y)[\xi_a, \xi_\beta].
\end{aligned}$$

Since Σ is normal, then in virtue of Theoreme 3 we have:

$$\overset{1}{N} = \overset{2}{N}{}^i = \overset{3}{N}_i = \overset{4}{N}{}^j_i = [\xi_i, \xi_j] = 0\,, \quad i,j = 1,2,\dots,r,$$

and hence $\overset{1}{N}{}^{\Sigma_1} = 0$ or Σ_1 is normal. In the similar way we can prove that the structure $\Sigma_2 = (\phi + \eta^a \otimes \xi_a, \eta, \xi)$ is also an almost paracontact structure. Let $q \in (0, r)$ be a positive integer and put

$$\Phi^q = \phi - \xi_1 \otimes \eta^1 - \xi_2 \otimes \eta^2 - \dots - \eta^{r-q}\xi_{r-q}\,,$$

then

$$\Sigma^q = (\Phi^q, \xi_{r-q+1}, \dots, \xi_r, \eta^{r-q+1}, \dots, \eta^r)$$

is an almost q-paracontact structure.

Let

$$D^{+q} = \{X; \Phi^q(X) = X\}, \quad D^{-q} = \{X; \Phi(X) = -X\}, \quad D^{0q} = \{X; \Phi^q(X) = 0\}.$$

Then we have:

$$\begin{aligned} D^{+q} &= D^+ , \\ D^{-q} &= D^- \oplus \text{ lin } \{\xi_1, \ldots, \xi_{r-q}\} , \\ D^0 &= \text{ lin } \{\xi_{r-q+1}, \ldots, \xi_r\} , \\ D^{+q} \oplus D^{0q} &= D^+ \oplus \text{ lin } \{\xi_{r-q+1}, \ldots, \xi_r\} , \\ D^{-q} \oplus D^{0q} &= D^- \oplus D^0 \end{aligned}$$

where D^+, D^-, D^0 are defined by (36) and lin $\{X_1, \ldots, X_r\}$ denotes a linear space being spanned by vector fields $X_1, \ldots, X_r$. Now we have the following:

Proposition 1. *If an almost r-paracontact structure* $\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\ldots,r}$ *on M is normal then an almost q-paracontact structure*

$$\Sigma^a = (\phi^q, \xi_{r-q+1}, \ldots, \xi_r, \eta^{r-q+1}, \ldots, \eta^r)$$

is also normal.

Proof. If Σ is normal, then D^+, D^-, $D^+ \oplus D^0$, $D^- \oplus D^0$ are integrable and $\overset{4}{N}{}^i_j = 0$ and $[\xi_i, \xi_j] = 0, i, j = 1, 2, \ldots, r$. From $\overset{4}{N}{}^i_j = 0$ we have $\alpha_{\xi_j}\eta^i = 0$ or $\xi_j\eta^i(X) - \eta^i[\xi_j, X] = 0$. For $X \in D^+$ and $X \in D^-$ we have $\eta^i[\xi_j, X] = 0$. Now from integrability of $D^+ \oplus D^0$ (resp. $D^- \oplus D^0$) in virtue of Lemma 3, for $X \in D^+$ (resp. $X \in D^-$), $[X, \xi_i] \in D^+ \oplus D^0$ (resp. $[X, \xi_i] \in D^- \oplus D^0$). On account of Lemma 2 and the condition $\eta^i[\xi_j, X] = 0$, we have for $X \in D^+$, $\phi[X, \xi_i] = [X, \xi_i]$ what means that $[X, \xi_i] \in D^+$ and similarly, for $X \in D^-$, $[X, \xi_i] \in D^-$. Hence the distributions D^{+q}, D^{-q}, $D^{+q} \oplus D^{0q}$, $D^{-q} \oplus D^{0q}$ are integrable and obviously $\overset{4}{N}{}^a_\beta = 0$, $[\xi_\alpha, \xi_\beta] = 0$, $\alpha, \beta = 1, \ldots, r-1$. Thus Σ^q is normal.

More general, put $\psi^q = \phi + \varepsilon_1\eta^1 \otimes \xi_1 + \varepsilon_2\eta^2 \otimes \xi_2 + \cdots + \varepsilon_{r-q}\eta^{r-q} \otimes \xi_{r-q}$ where $\varepsilon_\alpha = \pm 1$, $\alpha = 1, \ldots, r-q$. We have:

Theorem 10. $\Sigma^q = (\psi^q, \xi_{r-q+1}, \ldots, \xi_r, \eta^{r-q+1}, \ldots, \eta^r)$ *is an almost q-paracontact structure on M and if an almost r-paracontact structure*

$$\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\ldots,r}$$

on M is normal then Σ^q *is normal as well.*

Proof is similar to the one of Proposition 1.

Remark 4. Note that in the case $q = 1$ we get a geometric proof of Theorem 8. The proof of the above theorem one can do with an algebraic method (comp. Theorem 8), but it is more complicated than the one presented in Proposition 1.

Now if we put $\Phi = \varphi + \varepsilon_1\eta^1 \otimes \xi_1 + \varepsilon_2\eta^2 \otimes \xi_2 + \cdots + \varepsilon_r\eta^r \otimes \xi_r$, $\varepsilon_i = \pm 1$, then we obtain the tensor field of an almost product structure on M and we have:

Theorem 11. *If an almost r-paracontact structure*

$$\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}$$

on M is normal then an almost product structure Φ is integrable.

Proof. Without loss of generality, we may assume that $\varepsilon_1 = \cdots = \varepsilon_q = 1$ and $\varepsilon_{q+1}, \dots, \varepsilon_r = -1$. Then the distributions $D^{+\Phi}$ and $D^{-\Phi}$ are as follows:

$$D^{+\Phi} = \{X; \Phi X = X\} = D^{+} \oplus lin\{\xi_1, \dots, \xi_q\}$$

$$D^{-\Phi} = \{X; \Phi X = -X\} = D^{-} \oplus lin\{\xi_{q+1}, \dots, \xi_r\}.$$

Both distributions are integrable and in virtue of Remark 3 Φ is integrable.

Remark 5. Note that if a positive definite Riemannian metric g is compatible with an almost r-paracontact structure Σ, then g is also compatible with any almost q-paracontact structure Σ^q and an almost product structure Φ.

Examples. 1. Let $(\psi, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}$ be an almost r-contact structure [5] on M. Then $(\psi^2, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}$ is an almost r-paracontact structure on M.

2. Suppose that (M,g) is a Riemannian manifold and $\xi_1, \dots, \xi_r$ are orthonormal vector fields . Put $\eta^i(X) = g(X, \xi_i)$ and $\phi = \text{Id} - \eta^i \otimes \xi_i$. Then $\Sigma = (\phi, \xi_{(i)}, \eta^{(i)})_{i=1,\dots,r}$ is an almost r-paracontact structure on M and is called an almost r-paracontact ξ_i-structure. In this case:

$$D^{+\Sigma} = \{X; \eta^i(X) = 0, i = 1, \dots, r\},$$

$$D^{-\Sigma} = 0,$$

$$D^{0\Sigma} = \text{lin}\{\xi_1, \xi_2, \dots, \xi_r\}.$$

Now we prove the following:

Theorem 12. *An almost r-paracontact ξ_i-structure Σ on M is normal if and only if $d\eta^i = 0$ and $[\xi_i, \xi_j] = 0$, $i, j = 1, \dots, r$.*

Proof. Suppose that Σ is normal. If $X, Y \in D^{+\Sigma}$ then $[X, Y] \in D^{+\Sigma}$ and $d\eta^i(X, Y) = 0$. If $X \in D^{+\Sigma}$ then $d\eta^i(X, \xi_j) = -\xi_j \eta^i(X) - \eta^i[X, \xi_j] = -\overset{4}{N}{}^i_j(X) = 0$ and so $d\eta^i = 0$, $i = 1, 2, \dots, r$. Now, conversely, suppose that $d\eta^i = 0$, and $[\xi_i, \xi_j] = 0$, $i = 1, 2, \dots, r$. Let $X, Y \in D^{+\Sigma}$. Then we have $0 = d\eta^i(X, Y) = -\eta^i[X, Y]$. Thus $[X, Y] \in D^{+\Sigma}$. If $X \in D^{+\Sigma}$ then $0 = d\eta^i(X, \xi_j) = -\eta^i[X, \xi_j]$ and so $[X, \xi_j] \in D^{+\Sigma}$. Hence we have that the distributions $D^{+\Sigma}$, $D^{-\Sigma}$, $D^{+\Sigma} \oplus D^{0\Sigma}$, $D^{-\Sigma} \oplus D^{0\Sigma}$ are integrable and $\overset{4}{N}{}^i_j = 0$ and $[\xi_i, \xi_j] = 0, i, j = 1, 2, \dots, r$.

3. If we lift an almost paracontact structure $\Sigma = (\phi, \xi, \eta)$ on a manifold M to the tangent bundle TM we obtain, in the natural way, an almost 2-paracontact structure $\overline{\Sigma}$ on TM. Let $\Sigma = (\phi, \xi, \eta)$ be an almost paracontact structure on M. Then $\overline{\Sigma} = (\phi^c, \xi^c, \xi^v, \eta^v, \eta^c)$ is an almost 2-paracontact structure on TM, where ϕ^c, ξ^c, η^c are complete lifts and ξ^v, η^v are vertical lifts of ϕ, ξ, η, to the tangent bundle TM. Making use of the properties of complete and vertical lifts [6], we

have:

$$\overset{1}{N}{}^{\overline{\Sigma}_1}(X^c,Y^c) = [\phi^c X^c,\phi^c Y^c] + [X^c,Y^c] - \phi^c[\phi^c X^c,Y^c] - \phi^c[X^c,\phi^c Y^c] - 2d\eta^v(X^c,Y^c)\xi^v - 2d\eta^c(X^c,Y^c)\xi^v - \eta^v(X^c,Y^c)\xi^v - \eta^c(X^c,Y^c)\xi^v([\phi X,\phi Y] + [X,Y] - \phi[\phi X,Y] - \phi[X,\phi Y] - 2d\eta(X,Y)\xi - \eta[X,Y]\xi)^c = (\overset{1}{N}(X,Y))^c .$$

In virtue of Proposition 1 p.33 [6] we have:

Theorem 13. *An almost paracontact structure $\Sigma = (\phi,\xi,\eta)$ on M is normal if and only if an almost 2-paracontact structure $\overline{\Sigma} = (\phi^c,\xi^v,\xi^c,\eta^v,\eta^c)$ on TM is normal.*

From Theorem 10 and 13 we have:

Theorem 14. *If $\Sigma = (\phi,\xi,\eta)$ is an almost paracontact structure on M, then $\overline{\Sigma_1} = (\phi^c + e\xi^v \otimes \eta^c, \xi^c, \eta^v)$, and $\overline{\Sigma_2} = (\phi^c + e\xi^c \otimes \eta^v, \xi^v, \eta^c)$, where $e = \pm 1$, are almost paracontact structures on TM. Moreover if Σ is normal, then $\overline{\Sigma_1}$ and $\overline{\Sigma_2}$ are normal.*

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STRESZCZENIE

W pracy rozpatrujemy struktury r-parakontaktowe, będące naturalnym uogólnieniem struktur parakontaktowych, wprowadzonych przez I.Sato. Definiujemy pojęcie normalności takich struktur i podajemy jej interpretację geometryczną. Podajemy także przykłady struktur r-parakontaktowych.

РЕЗЮМЕ

В данной работе рассматриваются r-параконтакные структуры, которые обобщьют параконтактные структуры введенные И.Сатом. Определяется понятие нормальной r-структуры вместе с её геометрической интерпретацией. Работу кончаем примерами.