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Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

B. PRUS

On the Existence of Some Strictly Convex Functionals

O istnieniu pewnych funkcjonałów ściśle wypukłych

О существовании некоторых строго выпуклых функционалов

1. Introduction. Let **X** be a real Banach space with the norm $\|\cdot\|$ and let $\Psi : \mathbf{X} \to \mathcal{R}$ be a functional such that

(i) There exists a function $\beta : \mathcal{R}_+ \to \mathcal{R}_+$, nondecreasing, continuous and satisfying:

$$(x_i \in \mathbf{X}, \|x_i\| \leqslant r \ (i = 1, 2; r > 0)) \Rightarrow |\Psi(x_1) - \Psi(x_2)| \leqslant \beta(r) \|x_1 - x_2\| .$$

(ii) For any $t \in (0, 1)$ and $x, y \in \mathbf{X}$

$$\Psi(tx + (1-t)y) \leqslant t\Psi(x) + (1-t)\Psi(y) - \Gamma(t, \|x - y\|),$$

where $\Gamma(t, s) := t\gamma_1((1-t)s) + (1-t)\gamma_1(ts)$ $(t \in [0, 1], s \geqslant 0)$, $\gamma_1(s) = \int_0^s \gamma(t)dt$ and a function $\gamma : \mathcal{R}_+ \to \mathcal{R}$ is continuous, $\gamma(t) \nearrow +\infty$ $(t \to +\infty)$, $\gamma(t) \searrow 0$ $(t \to 0)$.

Such functionals have been defined and investigated by T. Leżanski [1]. The results of [1] have been applied to a minimization of convex functionals [3].

The object of this paper is to study the problem of the existence of functionals, satisfying (i) and (ii). It is easy to see that if **X** is a real Hilbert space, then the functional $\Psi(x) = \|x\|^2$ $(x \in \mathbf{X})$ satisfies (i) and (ii). In Section 2, we prove that if **X** is a super-reflexive Banach space, then there exist an equivalent norm $|\cdot|$ and a constant $p \geqslant 2$ such that the functional $\Psi(x) = |x|^p$ satisfies (i) and (ii). In Section 3, we give an example of such a functional defined on the Sobolev space $W^{k,p}(G)$ $(p \geqslant 2)$. Namely, it is the functional $\Psi(x) = \|x\|^p_{k,p}$ $(x \in W^{k,p}(G))$ ($\|\cdot\|_{k,p}$ denotes the norm of $W^{k,p}(G)$).

2. The Strictly Convex Functionals on Super-reflexive Spaces. To prove the main result of this section we shall make use of the results obtained by G. Pisier [2].

Let $(\Omega, \mathcal{A}, P)$ be a measure space with $P(\Omega) = 1$.

Definition 2.1. A sequence $\{X_n\}_{n \geq 0}$ of Banach space valued $\mathcal{A}$-measurable functions $X_n : \Omega \to \mathbf{X}$ $(n = 0, 1, \ldots)$ is called a martingale if there exists an increasing sequence of σ-subalgebras $\{\mathcal{A}_n\}_{n \geq 0}$ of $\mathcal{A}$ such that for every $n = 0, 1, 2, \ldots$ the function X_n is $\mathcal{A}_n$- and $\mathcal{A}_{n-1}$-measurable, integrable on Ω (i.e. $\int_\Omega \| X_n(\omega) \| P(d\omega) < +\infty$) and

$$\int_A X_{n+1}\, dP = \int_A X_n\, dP \text{ for every } A \in \mathcal{A}_n .$$

For every martingale $\{X_n\}_{n \geq 0}$ we shall denote $\{dX_n\}_{n \geq 0}$ the sequence of increments of the martingale $\{X_n\}_{n \geq 0}$, i.e. $dX_n := X_n - X_{n-1}$ $(n \geq 1)$, $dX_0 := X_0$.

We say that a Banach space $\mathbf{Y}$ is finitely representable in a Banach space $\mathbf{X}$ if for every finite demensional subspace M of $\mathbf{Y}$ and every $\epsilon > 0$ there is a subspace N of $\mathbf{X}$ such that $d(M, N) \leq 1 + \epsilon$ ($d(M, N) := \inf \| T \| \, \| T^{-1} \|$, where T runs over all the isomorphisms from M onto N, with the convention $\inf \emptyset = +\infty$) (cf. [2]).

Definition 2.2. A Banach space $\mathbf{X}$ is called super-reflexive if all the Banach spaces, which are finitely representable in $\mathbf{X}$ are reflexive.

We shall make use of the following theorem:

Theorem 2.1. (G. Pisier [2]). *If a Banach space $\mathbf{X}$ is super-reflexive, ten there exist constants $p \geq 2$ and $c > 0$ such that for all $\mathbf{X}$-valued martingales $\{X_n\}_{n \geq 0}$*

$$(1) \qquad \int_\Omega \| X_0 \|^p + \sum_{n=1}^{\infty} \int_\Omega \| dX_n \|^p \leq C^p \sup_{n \geq 0} \int_\Omega \| X_n \|^p .$$

Set $\Omega = [0, 1]$ and let $\mathcal{A}$ be a σ-algebra of Lebesgue measurable subsets of Ω, P – the Lebesgue measure.

Theorem 2.2. *Let $\mathbf{X}$ be a Banach space with the norm $\| \cdot \|$ and let $2 \leq p < +\infty$. If there exists a constant $c > 0$ such that all $\mathbf{X}$-valued martingales $\{X_n\}_{n \geq 0}$ satisfy (1), then there exists a norm $| \cdot |$ such that*

$$(2) \qquad \| x \| \leq | x | \leq C \| x \| \text{ for each } x \in \mathbf{X},$$

$$(3) \quad | tx + (1-t)y |^p \leq t | x |^p + (1-t) | y |^p - \| x - y \|^p \left(t(1-t)^p + (1-t)t^p\right)$$

for any $t \in (0, 1)$ and $x, y \in \mathbf{X}$.

Proof. For $x \in \mathbf{X}$ set

$$(4) \qquad | x | := \inf \left\{ \left(C^p \sup_{n \geq 0} \int_\Omega \| X_n \|^p - \sum_{n=1}^{\infty} \int_\Omega \| dX_n \|^p \right)^{1/p} \right\},$$

where the infimum is taken over all X-valued martingales $\{X_n\}_{n\geq 0}$ such that $X_0 \equiv x$ and $\sup_{n\geq 0} \int_\Omega \| X_n \|^p < +\infty$. It follows from (1) that $| x | < +\infty$ and $\| x \| \leqslant | x |$ for all $x \in \mathrm{X}$. On the other hand, if we consider the martingale $\{X_n\}_{n\geq 0}$ such that $X_n \equiv x$ ($n = 0, 1, 2, \ldots$), we get $| x | \leqslant C \| x \|$ $(x \in \mathrm{X})$.

Let $x, y \in \mathrm{X}$. By the definition of $| x |$ and $| y |$, for all $\gamma > 0$ there exist martingales $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ such that

$$X_0 \equiv x, \sup_{n\geq 0} \int_\Omega \| X_n \|^p < +\infty$$

$$Y_0 \equiv y, \sup_{n\geq 0} \int_\Omega \| Y_n \|^p < +\infty \text{ and}$$

$$C^p \sup_{n\geq 0} \int_\Omega \| X_n \|^p - \sum_{n=1}^{\infty} \int_\Omega \| dX_n \|^p \leqslant | x |^p + \gamma, \tag{5}$$

$$C^p \sup_{n\geq 0} \int_\Omega \| Y_n \|^p - \sum_{n=1}^{\infty} \int_\Omega \| dY_n \|^p \leqslant | y |^p + \gamma. \tag{6}$$

Let $\{\mathcal{A}_n\}_{n\geq 0}$ denote an increasing sequence of σ-subalgebras of $\mathcal{A}$ relative to $\{X_n\}_{n\geq 0}$ and $\{\mathcal{B}_n\}_{n\geq 0}$ – relative to $\{Y_n\}_{n\geq 0}$ (cf. Definition 2.1).

Let us fix $t \in (0, 1)$. For $\omega \in \Omega = [0, 1]$ set

$$\varphi_1(\omega) := t\omega,$$

$$\varphi_2(\omega) := (1 - t)\,\omega + t.$$

Observe that the functions φ_1 and φ_2 are increasing (hence one-to-one) and that $\varphi_1(\Omega) = [0, t]$, $\varphi_2(\Omega) = [t, 1]$.

We define a new martingale $\{Z_n\}_{n\geq 0}$ setting

$$Z_0(\omega) := tx + (1 - t)y \text{ for } \omega \in \Omega,$$

$$Z_n(\omega) := \begin{cases} X_{n-1}(\varphi_1^{-1}(\omega)) \text{ for } \omega \in [0, t), \\ Y_{n-1}(\varphi_2^{-1}(\omega)) \text{ for } \omega \in [t, 1] \end{cases} \quad (n \geqslant 1).$$

Let $\mathcal{C}_n$ $(n \geqslant 0)$ be the σ-algebra of all sets the form

$$C_n = \varphi_1(A_{n-1} \cap [0, 1)) \cup \varphi_2(B_{n-1}) \quad (n \geqslant 1)$$

for some sets $A_{n-1} \in \mathcal{A}_{n-1}$, $B_{n-1} \in \mathcal{B}_{n-1}$; $\mathcal{C}_0 = \{\emptyset, \Omega\}$. It is not difficult to show that $\{Z_n\}_{n\geq 0}$ (with $\{\mathcal{C}_n\}_{n\geq 0}$) is a martingale (cf. Definition 2.1).

Observe that

$$\int_\Omega \| Z_n(\omega) \|^p \, d\omega = \int_0^t \| X_{n-1}(1/t\omega) \|^p \, d\omega + \int_t^1 \| Y_{n-1}\,(1/(1-t)\,(\omega - t)) \|^p \, d\omega =$$

$$= t\int_\Omega \| X_{n-1}(\omega)\|^p d\omega + (1-t)\int_\Omega \| Y_{n-1}(\omega)\|^p d\omega,$$

so $\sup\limits_{n\geq 0}\int_\Omega \| Z_n \|^p < +\infty$. Hence it follows from (4) (with X_n replaced by Z_n) that

$$|tx + (1-t)y|^p \leq C^p \sup_{n\geq 0}\int_\Omega \| Z_n \|^p - \sum_{n=1}^{\infty}\int_\Omega \| dZ_n \|^p. \tag{7}$$

We observe that

$$dZ_n(\omega) = \begin{cases} dX_{n-1}(\varphi_1^{-1}(\omega)) \text{ for } \omega \in [0,t), \\ dY_{n-1}(\varphi_2^{-1}(\omega)) \text{ for } \omega \in [t,1]\ (n \geq 2) \end{cases}$$

and hence

$$\sum_{n=2}^{\infty}\int_\Omega \| dZ_n \|^p = \sum_{n=1}^{\infty}\left(t\int_\Omega \| dX_n \|^p + (1-t)\int_\Omega \| dY_n \|^p\right).$$

Furthemore

$$dZ_1(\omega) = \begin{cases} (1-t)(x-y) \text{ for } \omega \in [0,t), \\ t(y-x) \quad\quad \text{ for } \omega \in [t,1], \end{cases}$$

so

$$\int_\Omega \| dZ_1(\omega)\|^p d\omega = \int_0^t \| (1-t)(x-y)\|^p d\omega + \int_t^1 \| t(y-x)\|^p d\omega =$$

$$= t(1-t)^p \| x-y \|^p + (1-t)t^p \| x-y \|^p.$$

In view of (7), (5) and (6) we finally obtain

$$|tx + (1-t)y|^p \leq C^p t \sup_{n\geq 0}\int_\Omega \| X_n \|^p + C^p (1-t)\sup_{n\geq 0}\int_\Omega \| Y_n \|^p -$$

$$-((1-t)^p t + (1-t)t^p)\| x-y \|^p - t\sum_{n=1}^{\infty}\int_\Omega \| dX_n \|^p - (1-t)\sum_{n=1}^{\infty}\int_\Omega \| dY_n \|^p \leq$$

$$\leq t|x|^p + (1-t)|y|^p - ((1-t)^p t + (1-t)t^p)\| x-y \|^p + 2\gamma.$$

Since $\gamma > 0$ is arbitrary, we obtain the inequality (3). Remark that (3) implies also that the function $x \to |x|$ satisfies the triangle inequality, so it is a norm on $\mathbf{X}$. The proof is completed.

From Theorems 2.1 and 2.2 we obtain

Corollary 2.1. *If* $\mathbf{X}$ *is a super-reflexive Banach space with the norm* $\| \cdot \|$, *then there exists a functional* $\Psi : \mathbf{X} \to \mathcal{R}$ *satisfying* (i) *and* (ii). *For, put* $\Psi(x) := | x |^p$. *To prove* (i), *let* $x_1, x_2 \in \mathbf{X}$, $\| x_1 \| \leq r$, $\| x_2 \| \leq r$. *Then*

$$| \Psi(x_1) - \Psi(x_2) | \leq pC^{p-1} r^{p-1} | x_1 - x_2 | \leq p\, C^p\, r^{p-1} \| x_1 - x_2 \| .$$

The condition (ii) (*with* $\Gamma(t, s) = s^p (t (1-t)^p + t^p (1-t))$) *follows from* (3).

3. The Example of the Functional Ψ on the Sobolev Space $W^{k,p}(G)$ ($p \geq 2$).
For $\tau \in \mathcal{R}$ and $t \in (0, 1)$ set

(8) $f(\tau, t) := (t (1-t)^p + t^p (1-t))^{-1} (t + (1-t) | \tau |^p - | t + (1-t) \tau |^p)$ $(p \geq 2)$.

First we prove the following

Lemma 3.1. *If* $p \geq 2$, *then there exists* $c > 0$ *such that*

(9) $$f(\tau, t) / (1-\tau)^p \geq c \textit{ for all } \tau \in [-1, 1) \textit{ and } t \in (0, 1).$$

Proof. Observe that $f(\tau, t) \geq 0$ ($\tau \in \mathcal{R}$, $t \in (0, 1)$) and that for $t \in (0, 1)$ fixed the function $\tau \to f(\tau, t)$ is differentiable and

$$f(1, t) = d/d\tau|_{\tau=1} f(\tau, t) = 0,$$

$$d^2/d\tau^2|_{\tau=1} f(\tau, t) = (t (1-t)^p + t^p (1-t))^{-1}\, p (p-1) (1-t)\, t,$$

$$d^k/d\tau^k|_{\tau=1} f(\tau, t) \geq 0 \ (k \geq 2).$$

Therefore, for $t \in (0, 1)$ fixed

$$f(\tau, t) = \frac{p (p-1)}{2 ((1-t)^{p-1} + t^{p-1})} (\tau - 1)^2 + \frac{d^3/d\tau^3|_{\tau=1} f(\tau, t)}{3!} (\tau - 1)^3 + \ldots (\tau > 0).$$

Hence, if $\tau \in (1, 2]$, then

$$\frac{f(\tau, t)}{(\tau - 1)^p} \geq \frac{f(\tau, t)}{(\tau - 1)^2} \geq \frac{p (p-1)}{2 ((1-t)^{p-1} + t^{p-1})} \geq \frac{p (p-1)}{2} .$$

Let now $\tau \in [1/2, 1)$; then

$$\frac{\tau^p f(1/\tau, t)}{(1-\tau)^p} \geqslant \frac{p(p-1)}{2}.$$

But, since $\tau^p f(1/\tau, t) = f(\tau, 1-t)$,

$$\frac{f(\tau, t)}{(1-\tau)^p} \geqslant \frac{p(p-1)}{2} \tag{10}$$

for any $\tau \in [½, 1)$ and $t \in (0, 1)$.

Now if $\tau \in [-1, 1/2]$, then $(1-\tau)^p \leqslant 2^p$. Therefore it suffices to show that there exists $\tilde{c} > 0$ such that $f(\tau, t) \geqslant \tilde{c}$ for any $t \in (0, 1)$. Observe that the function $\tau \to f(\tau, t)$ is nonincreasing on each of the intervals $(-1, t/(t-1)), (t/(t-1), 0))$ $(t \in (0, 1/2))$ and $(0, 1/2)$ (because of $(d/d\tau) f(\tau, t) \leqslant 0$ on these intervals). Hence

$$f(\tau, t) \geqslant f(1/2, t) \; (\tau \in [-1, 1/2], t \in (0, 1)).$$

We have

$$f(1/2, t) = \frac{1 - (1/2^p)(t^{p-1} + \ldots + \binom{p}{p-1} + 1)}{t^{p-1}(1-t) + (1-t)^p},$$

$$\lim_{t \searrow 0} f(1/2, t) = 1 - (p+1)/2^p > 0, \lim_{t \nearrow 1} f(1/2, t) = 1/2^p \left((p-1) + \ldots + \binom{p}{p-2}\right) > 0.$$

If $\inf_{t \in (0,1)} f(1/2, t) = 0$, there exists $t_0 \in (0, 1)$ such that $f(1/2, t_0) = 0$. But $1/2^p (t^{p-1} + \ldots + \binom{p}{p-1} + 1) < 1$ for all $t \in (0, 1)$, i.e. $f(1/2, t_0) > 0$. Hence $\inf_{t \in (0,1)} f(1/2, t) > 0$, so there exists $\tilde{c} > 0$ such that $f(1/2, t) \geqslant \tilde{c}$ for all $t \in (0, 1)$.

Therefore, for $\tau \in [-1, 1/2]$ and $t \in (0, 1)$ we have

$$\frac{f(\tau, t)}{(1-\tau)^p} \geqslant \frac{\tilde{c}}{2^p}$$

and by virtue of (10)

$$\frac{f(\tau, t)}{(1-\tau)^p} \geqslant C \text{ for } \tau \in [-1, 1) \text{ and } t \in (0, 1),$$

where $C = \min \left\{ p(p-1)/2, \tilde{c}/2^p \right\}$. The proof is complete.

Let $G \subset \mathbb{R}^n$ denote an open set. For $k \in \mathrm{N} \cup \{0\}$ and $p \geqslant 1$ we denote by $W^{k,p}(G)$ the set of all functions $x : \xi = (\xi_1, \ldots, \xi_n) \ni G \to \mathbb{R}$ such that x and its distributional derivatives $D^s x = (\partial^{|s|} x) / (\partial \xi_1^{s_1} \partial \xi_2^{s_2} \ldots \partial \xi_n^{s_n})$ $(s = (s_1, \ldots, s_n))$ of order $|s| = \sum_{j=1}^{n} |s_j| \leqslant k$ all belong to $L^p(G)$. $W^{k,p}(G)$ is a Banach space by the norm

$$\| x \|_{k,p} := \Big(\sum_{|s| \leqslant k} \int_G | D^s x(\xi) |^p \, d\xi \Big)^{1/p}.$$

Now, we can show

Theorem 3.1. *If $p \geqslant 2$, then there exists a constant $C > 0$ such that for all $x, y \in$ $\in W^{k,p}(G)$ and $t \in (0,1)$*

$$\text{(11)} \qquad \| tx + (1-t) y \|_{k,p}^p \leqslant t \| x \|_{k,p}^p + (1-t) \| y \|_{k,p}^p -$$
$$- C \| x - y \|_{k,p}^p \, (t (1-t)^p + t^p (1-t)).$$

Proof. First note that, in view of Lemma 3.1 the inequality

$$\text{(12)} \quad | t + (1-t) \tau |^p \leqslant t + (1-t) | \tau |^p - C | 1 - \tau |^p \, (t (1-t)^p + t^p (1-t))$$

holds for all $\tau \in [-1, 1)$ and $t \in (0, 1)$. It is easy to see that (12) holds also for $\tau = 1$.

Let $\alpha, \beta \in \mathbb{R}$. Assume that $\alpha \neq 0$ and $|\beta/\alpha| \leqslant 1$ and put $\tau = \beta/\alpha$ in (12) (if $|\beta/\alpha| > 1$ or $\alpha = 0$ we put $\tau = \alpha/\beta$). Therefore, we have

$$| t\alpha + (1-t) \beta |^p \leqslant t |\alpha|^p + (1-t) |\beta|^p - C |\alpha - \beta|^p (t (1-t)^p +$$
$$+ t^p (1-t)) \; (|\beta/\alpha| \leqslant 1, \alpha \neq 0, t \in (0, 1)),$$

$$| t\beta + (1-t) \alpha |^p \leqslant t |\beta|^p + (1-t) |\alpha|^p - C |\beta - \alpha|^p (t (1-t)^p +$$
$$+ t^p (1-t)) \; (|\beta/\alpha| > 1 \text{ or } \alpha = 0, t \in (0, 1)).$$

The above inequalities hold for any $t \in (0, 1)$, hence for all $\alpha, \beta \in \mathbb{R}$ and $t \in (0, 1)$ we have

$$\text{(13)} \quad | t\alpha + (1-t) \beta |^p \leqslant t |\alpha|^p + (1-t) |\beta|^p - C |\alpha - \beta|^p (t (1-t)^p + t^p (1-t)).$$

It is easy to see that (13) implies (11), so the proof is complete.

Corollary 3.1. *The functional $\Psi(x) = \| x \|_{k,p}^p$ satisfies* (i) *and* (ii) (*with* $\Gamma(t, s) =$ $= Cs^p (t (1-t)^p + t^p (1-t))$ (cf. Corollary 2.1).

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STRESZCZENIE

W pracy zajmujemy się problemem istnienia funkcjonału $\Psi : X \to R$ (X – rzeczywista przestrzeń Banacha), spełniającego warunki (i) oraz (ii). W rozdziale 2 wykazujemy, że jeśli przestrzeń X jest super-refleksywna, to istnieje norma $|\cdot|$ równoważna normie wyjściowej przestrzeni X oraz stała $p \geq 2$ taka, że funkcjonał $\Psi(x) = |x|^p$ spełnia wyżej wymienione warunki. W rozdziale 3 podajemy przykład takiego funkcjonału określonego na przestrzeni Sobolewa $W^{k,p}(G)$ ($k \in N \cup \{0\}$, $p \geq 2$). Dowodzimy, że $\Psi(x) = \|x\|_{k,p}^p$ ($\|\cdot\|_{k,p}$ oznacza normę przestrzeni $W^{k,p}(G)$) spełnia warunki (i) oraz (ii).

РЕЗЮМЕ

В работе преднтавлено проблему существования функционала $\Psi : X \to R$ (X – вещественное банахового продстранства удовлетворяющее условиям (i) и (ii). Во второй части доказывается что если пространство X супер-рефлексивно, то существует норма $|\cdot|$ эквивалентна исходной норме пространства X и постоянная $p \geq 2$ такие, что функционал $\Psi(x) = |x|^p$ удовлетворяет этим условиям. В части 3 построен пример такого функционала, заданного на пространстве Соболева $W^{k,p}(G)$ ($k \in N \cup \{0\}$, $p \geq 2$). Доказывается, что $\Psi(x) = \|x\|_{k,p}^p$ (где $\|\cdot\|_{k,p}$ норма пространства $W^{k,p}(G)$) удовлетворяет условиям (i) и (ii).