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Bielecki's Method in the Theory of Integral Equations

Metoda Bieleckiego w teorii równań całkowych

Метод Белецкого в теории интегральных уравнений

Introduction. In 1956, A. Bielecki has published two notes in the Bulletin de l'Academie Polonaise des Sciences (see [7] and [8]), in which a new procedure of obtaining global existence results for differential equations (ordinary, or with partial derivatives) is illustrated.

This new procedure, according to the statement made in [7], was aimed at improving the estimate of the length of the interval of existence of solutions, by choosing in the underlying space a norm with respect to which the attached operator becomes a contraction. Thus, besides existence and uniqueness of solution, the Bielecki's procedure in applying the fixed point theorem leads to a wider domain of definition for the solution (than in case of a "crude" application of the same principle), and – simultaneously – to an estimate for the solution in terms of the data. This last feature better enhanced when the interval of existence is not a compact one, and it is illustrated (for instance) in the author's paper [12].

Bielecki's method has been applied, since its inception in [7] and [8], by many authors. It has become such a common tool in handling existence and uniqueness of solutions of equations (differential, differential-difference, integral), that some authors do not feel compelled anymore to quote the originator.

Without any attempt to be complete in regard to the existing literature on this subject, we would like to mention some papers in which the Bielecki's method has been illustrated on various classes of equations, and under various assumptions.

For instance, in case of ordinary differential equations, the method has been used by the author in [12] and [13].

* Some results in this paper made the object of a lecture given by the author at the Mathematical Institute of the Lublin University in September 1975.

D. Petrovanu [35] has used the Bielecki's method in proving basic results for Pfaff's system.

For equations with delay, the method was applied in the author's paper [14], by M. C. Delfour and S. K. Mitter [23], S. Czerwik [22], and others. Recently, M. C. Delfour [24] applied the method in case of equations of the form

$$\dot{x}(t) = \int_{-\infty}^{0} [A(s)\, x(t+s) + B(s)\, \dot{x}(t+s)]\, ds,$$

under the main assumption $\| A(s) \|, \| B(s) \| \in L^q(-\infty, 0)$. The solution is sought in the Sobolev space $W^{1,p}(0, T), T > 0$, where $p^{-1} + q^{-1} = 1$.

In the case of integrodifferential equations with Volterra operators on the right hand side, Bielecki's method has been used by the author [16], by T. Talpalaru [39], and by M. Turinici [41].

Many more papers on the subject, but dealing with integral or integrofunctional equations, have been published during the last 20 years. It is our aim to survey in this paper the contributions falling into this category, and to emphasize the developments Bielecki's method had generated.

The Case of Volterra Integral Equations. We shall briefly illustrate, in this section, the use of Bielecki's method in the case of Volterra equations. We are not aiming here at the greatest generality, because forthcoming sections of this paper will be dedicated to recent contributions, emphasizing more general setting-ups for the problem.

Let us consider the nonlinear Volterra integral equation

$$\text{(E)} \qquad x(t) = f(t) + \int_0^t k(t, s, x(s))\, ds, \quad t \geq 0,$$

where x, f and k take their values in R^n. We assume that f is a continuous function on $R_+ = \{t \mid t \geq 0\}$, while $k(t, s, x)$ is also continuous on $\Delta \times R^n$, where

$$\text{(1)} \qquad \Delta = \{(t, s) \mid 0 \leq s \leq t\}.$$

Moreover, we assume $k(t, s, 0) \equiv 0$ on Δ, a condition which can be achieved substituting $k(t, s, x) - k(t, s, 0)$ to $k(t, s, x)$ in (E), and $f(t) + \int_0^t k(t, s, 0)\, ds$ to $f(t)$.

The following generalized Lipschitz condition is imposed on k:

$$\text{(2)} \qquad \| k(t, s, x) - k(t, s, y) \| \leq k_0(t, s) \| x - y \|,$$

where $\| \cdot \|$ denoted the norm in R^n, and $k_0(t, s)$ is a nonnegative continuous function on Δ.

Of course, under above stated conditions, one expects existence and uniqueness of the continuous solution to (E), defined on R_+.

In order, to carry out the proof, let us look first for a convenient space (of continuous functions from R_+ into R^n) in which the solutions is to be sought. Assuming $x(t)$ is a continuous solution of (E) defined on R_+, one easily derives the existence of a continuous positive function $g(t)$, also such that $\| x(t) \| \leqslant g(t)$, $t \in R_+$. We shall determine one of the possible choices for $g(t)$.

Let us consider now the Banach space of continuous functions $x(t)$, from R_+ into R^n, such that

(3) $\quad \| x(t) \| \leqslant M_x g(t), \quad t \in R_+,$

with $0 \leqslant M_x \leqslant +\infty$. If one denotes this space with the norm

(4) $\quad | x |_{C_g} = \sup \dfrac{\| x(t) \|}{g(t)}, \quad t \in R_+,$

by $C_g(R_+, R^n)$, it is known [15], [21] that C_g is complete (i.e., a Banach space).

Now, for some convenient g, we want to apply the Banach fixed point theorem to obtain existence and uniqueness for equation (E) in the space $C_g(R_+, R^n)$.

Let us denote

(5) $\quad (Tx)(t) = f(t) + \int_0^t k(t, s, x(s))\, ds, \quad t \in R_+,$

for any $x \in C_g$. It is obvious that $(Tx)(t)$ is a continuous function on R_+, with valucs in R^n. But $(Tx)(t)$ does not necessarily belong to C_g. Since $x(t) \equiv \theta$ (the null element in R^n) does belong to any C_g, one derives from (5) that $f(t)$ must belong to C_g, in order to achieve $TC_q \subset C_g$. Furthermore, the condition

(6) $\quad \int_0^t k(t, s, x(s))\, ds \in C_g, \quad x \in C_g,$

will be guaranted by

(7) $\quad \int_0^t k_0(t, s) g(s)\, ds \leqslant \alpha g(t), \quad t \in R_+,$

with α a positive constant. Indeed, from our assumptions one obtains only

(8) $\quad \| \int_0^t k(t, s, x(s))\, ds \| \leqslant \int_0^t k_0(t, s) \| x(s) \|\, ds, \quad t \in R_+.$

From (8) and (7), taking into account the definition of C_g, one derives the validity of (5).

Hence, assuming only $f \in C_g$, and the condition (7), besides other conditions on $k(t, s, x)$ formulated in this section, one obtains the inclusion

(9) $\quad TC_g \subset C_g$

It remains now to prove the existence of a function g, such that the inclusion (9) is a contraction. Actually, this is the key of Bielecki's method.

From (5), (7), and the definition of the norm in C_g, one obtains

(10) $\quad \| (Tx)(t) - (Ty)(t) \| \leqslant \int_0^t k_0(t, s) \dfrac{\| x(s) - y(s) \|}{g(s)} g(s)\, ds \leqslant$

$$\leq |x-y|_{C_g} \int_0^t k_0(t,s) g(s)\, ds \leq \alpha |x-y|_{C_g}\, g(t),\ t \in R_+,$$

which obviously implies

(11) $\quad |Tx - Ty|_{C_g} \leq \alpha |x-y|_{C_g}$.

Therefore, T is a contraction any time $0 \leq \alpha < 1$ in (7), and this implies the existence and uniqueness of the solution in C_g.

Of course, the existence of a function $g(t)$, satisfying (7), has yet to be proved. Let us remark first that (7) is implied by the inequality

(12) $\quad \tilde{k}(t) \int_0^t g(s)\, ds \leq \alpha g(t),\ t \in R_+,$

where $\tilde{k}(t)$ is a positive continuous function verifying

(13) $\quad k_0(t,s) \leq \tilde{k}(t),\ 0 \leq s \leq t.$

In order to construct $k(t)$, one can proceed as follows: let $\bar{k}(r) = \sup k_0(t,s)$, $0 \leq s \leq$ $\leq t \leq r$, $r \geq 0$. Then set $\tilde{k}(t) = \int_t^{t+1} \bar{k}(r)\, dr$, $t \geq 0$.

A solution of the inequality (12) is obviously given by

(14) $\quad g(t) = \tilde{k}(t) \exp\left\{\alpha^{-1} \int_0^t k(s)\, ds\right\},\ t \in R_+$

which is, basically, the function used by Bielecki in [7] in constructing his norm.

Therefore, we can state the following result in regard to the integral equations (E):

Theorem 1. *Consider the equation* (E) *under the following assumptions:*

a) *k is a continuous map from $\Delta \times R^n$ into R^n, such that the generalized Lipschitz condition* (2) *holds true, with $k_0 : \Delta \to R_+$ continuous;*

b) *$f \in C_g(R_+, R^n)$, where g is defined by* (14), *and the norm in C_g by* (4)*;*

c) *the constant α in* (14) *is such that $0 < \alpha < 1$.*

Then, there exists in the space C_g a unique solution of the equation (E), *which is the unique fixed point of the operator T defined by* (5).

Remark 1. In Bielecki's paper [7] it is assumed that $\tilde{k}(t)$ dominates also $\|f(t)\|$. Of course, this is no restriction as long as the underlying space is $C_g(I, R^n)$ with I a compact interval of R. Since R_+ (or any interval $[0, T)$, $T > 0$) is not compact, the assumption (b) in Theorem 1 allows more generality than $\|f(t)\| \leq \tilde{k}(t)$.

Remark 2. The uniqueness in Theorem 1 is stated only with respect to the class C_g. In order to obtain the uniqueness in the class of all continuous functions from R_+ into R^n, one can rely on the following argument.

Any continuous function from R_+ into R^n belongs to some $C_g(R_+, R^n)$. If we assume (E) has more than one solution, say $x \in C_g$ and $y \in C_h$, with $x \neq y$, then considering the function $m(t) = \max\{g(t), h(t)\}$, $t \in R_+$, one has $C_m \subset C_g \cup C_h$. But $m(t)$ satisfies the inequality (12) if both $g(t)$ and $h(t)$ satisfy that inequality. This means $TC_m \subset C_m$ for $0 < \alpha < 1$, which ends the proof.

Function Spaces. From the discussion conducted above in proving Theorem 1, it appears clearly that the choice of the underlying space (or of the norm conducing at that space) plays a significant role in obtaining adequate results, with global character, for integral equations of Volterra type.

The space C_g defined in the preceding section has been considered by the author in [15], in connection with the existence of solutions to integral equations of Hammerstein--Volterra type. This space has been investigated in [15], [18], in regard to the continuity of certain linear integral operators of Volterra type, as well as in regard to some Fredholm's type operators.

The space C_g have been used by many authors in the framework of Volterra integral equations: C. Avramescu [1], [2], G. Bantas [3] – [6], G. Dotseth [25], N. Pavel [34], D. Petrovanu [36], O. Staffans [37], [38], P. Talpalaru [39], M. Turinici [40], [41], V. A. Tyshkevich [42], L. B. Tzalyuk [43]. The monograph by V. A. Tyshkevich [42], and the survey paper by L. B. Tzalyuk [43] contain long lists of references pertinent to the use of C_g spaces in various problems of existence for integral equations. The books by V. Lakshmikantham and S. Leela [28], and by R. K. Miller [29], also contain various results and references related to C_g spaces. Finally, in connection with the use of C_g spaces in probability problems, we refer the reader to the books by T. A. Barucha-Reid [10], and by W. J. Padgett and C. P. Tsokos [32]. More references in respect to the spaces of random variables, of type C_g, are included in these two books.

More significant, perhaps, than the direct use of the C_g spaces in problems related to integral equations is the fact that several interesting generalizations of these spaces have been considered by the researchers.

First, we shall indicate the generalization due to H. E. Gollwitzer [27]. It deals with C_g spaces for which g is a matrix valued function, instead of a scalar function. Moreover, by means of the concept of a generalized inverse (to a matrix, in this case), Gollwitzer covers the case when $g = g(t)$ is not invertible for some $t \in R_+$ (generalized inverse of a matrix).

In order to formulate the definition of C_g spaces introduced and investigated by Golwitzer, we need some preparation concerning the concept of a generalized inverse of a square matrix.

Let $\mathbf{L}(R^n, R^n)$ be the linear space of all linear maps from R^n into itself. If an orthogonal basis of R^n is given, then $\mathbf{L}(R^n, R^n)$ coincides with the class of all n by n matrices with real entries. For any $A \in \mathbf{L}(R^n, R^n)$, let $\mathbf{N}_A$ and $\mathbf{R}_A$ be respectively the kernel (null space) and the range of A. Denote by $\mathbf{P}_A$ the orthogonal projector of R^n onto $\mathbf{R}_A$, and let $\widetilde{A}$ be the restriction of A to the orthogonal complement (in R^n) of $\mathbf{N}_A$, say $N_A^{\perp}$. Obviously, $\widetilde{A}$ is invertible on $\mathbf{R}_A = \mathbf{R}_{\widetilde{A}}$. We now define

$$(15)\quad A_{-1} = \widetilde{A}^{-1}\mathbf{P}_A,\ A \in \mathbf{L}(R^n, R^n),$$

which implies $A_{-1} : R^n \to N_A^{\perp}$. From (15) one derives $A_{-1}Ax = x$, for every $x \in N_A^{\perp}$. In case A is a nonsingular matrix, $N_A = \{0\}$, $N_A^{\perp} = R^n$, which means $A_{-1} = A^{-1}$.

Assume now $g : R_+ \to L(R^n, R^n)$ is a continuous map (the continuity is not necessary), and let us keep the notations used above.

The space $C_g(R_+, R^n)$ is defined by means of the following properties:

1) $x \in C_g$ is continuous;

2) $P_{g(t)}\,x(t) = x(t), \forall t \in R_+$;

3) $g_{-1}(t)\,x(t)$ is bounded on R_+.

The norm in C_g is defined by

(16) $|x|_g = \sup\left\{ \| g_{-1}(t)x(t) \|, \ \ t \in R_+ \right\}$.

As shown in [27], C_g is a Banach space whose topology is stronger than the topology of uniform convergence on any compact interval of R_+.

When $g(t)$ stands for the matrix $g(t)\,I$, where $g(t)$ is a continuous scalar function, the C_g defined by Gollwitzer reduces to the space C_g defined by the author in [15]. Of course, the spaces defined by Gollwitzer present much more flexibility in regard to the behaviour at infinity of the functions belonging to them. In particular, choosing $g(t) = \text{diag}\left\{ g_1(t), g_2(t), \ldots, g_n(t) \right\}$, with $g_i(t)$ continuous and positive on R_+, one can measure the growth of each component separately (having, for instance, some of them bounded on R_+, some tending to zero at infinity, etc.).

If insteated of continuous maps we are interested in (Lebesque) measurable ones, the scheme above leads easily to the definition of the spaces $L_g^{\infty}(R_+, R^n)$. The case of a scalar g is discussed in [36], while the case $g \in L(R^n, R^n)$ is treated in [27]. The key point, in this case, is the fact that $g_{-1}(t)$ is measurable, anytime $g(t)$ is. See [27] for details.

Finally, the scheme of constructing new function spaces proposed by Gollwitzer [27] can be applied to define the $L_g^p(R_+, R^n)$ spaces, $1 \leqslant p < \infty$, with $g : R_+ \to L(R^n, R^n)$ a measurable map. This case has been dealt with in [25] by G. M. Dotseth. Briefly, the space $L_g^p(R_+, R^n)$ consists of those measurable maps $x(t)$ for which $P_{g(t)}x(t) = x(t)$ a.e. on R_+, and $g_{-1}(t)\,x(t) \in L^p(R_+, R^n)$. The norm is given by

(17) $|x|_{L_g^p} = \left(\int_{R_+} \| g_{-1}(t)\,x(t) \|^p dt \right)^{1/p}$.

In relationship to the scheme divised by Gollwitzer, the following basic problem arises: when does the space $C_g(R_+, R^n)$ coincide with another space of the same type, say $C_h(R_+, R^n)$?

In case of scalar weight functions g and h it is obvious that the condition

(18) $ch(t) \leqslant g(t) \leqslant Ch(t), \ \forall t \in R_+$,

where c and C denote two positive constants, is a necesary and sufficient condition for having $C_g = C_h$.

In case of matrix valued g and h, the necessary and sufficient condition for the identity of Gollwitzer's spaces C_g and C_h has been obtained by L. Pandolfi [33]:

Theorem 2. *A system of necessary and sufficient conditions for the identity of the spaces C_g and C_h consists of:*

a) $\mathbf{R}_{g(t)} = \mathbf{R}_{h(t)}$, *for any* $t \in R_+$;

b) $\sup\left\{\|h_{-1}(t)\,g(t)\|;\, t\in R_+\right\} < +\infty$;

c) $\sup\left\{\|g_{-1}(t)\,h(t)\|;\, t\in R_+\right\} < +\infty$.

To the best of the author's knowledge, no similar results seems to be available in case of L^p_g spaces, $1 \leqslant p < \infty$. For $p = \infty$, the conditions are basically those in Theorem 2, with a.e. and ess-sup ingredients.

A further generalization of C_g (or L^p_g) spaces is due to M. Milman [30], [31]. If one examines the definitions formulated above (following Gollwitzer and Dotseth) for the spaces C_g or L^p_g, one sees that a condition of "boundedness" is always involved. This "boundedness" is understood in the sense of the supremum norm for C_g spaces, as essential-supremum for the spaces L^∞_g, or the usual L^p-norm, $1 \leqslant p < \infty$, for L^p_g.

Milman's idea consists in extending those definitions by using an abstract Banach space of functions, and to consider "boundedness" with respect to the norm of that function space. It is therefore useful to formulate the definition of a Banach function space. We have in mind functions defined on R_+, and taking values in R^n, the measure we are going to use is the Lebesque measure on R. This choice is consistent with the preceding definitions, but the scheme applies to more general situations [30], [31].

Let us first the Banach function spaces $S = S(R_+, R)$, consisting of locally integrable functions, i.e., $S \in L_{loc}(R_+, R)$, and whose norm $|\cdot|_S$ (the subscript will be ommitted) satisfies the following conditions:

1) If $y \in S$, and $x \in L_{loc}(R_+, R)$, then $|x(t)| \leqslant |y(t)|$ a.e. on R_+ implies $x \in S$, and $|x| \leqslant |y|$.

2) If $E \subset R_+$ is a measurable set, of finite (Lebesque) measure, then its characteristic function $\chi_E \in S$.

3) If $\{f_n\} \subset S$ is such that $0 \leqslant f_n(t) \uparrow f(t)$ a.e. on R_+, and $\{|f_n|\}$ is bounded, then $f \in S$, and $|f_n| \to |f|$ as $n \to \infty$.

It can be shown that 3) implies the completeness of S.

The Banach function space $S(R_+, R^n)$, is then defined by the property: $x \in S(R_+, R^n)$ iff each component $x_i \in S(R_+, R)$, $i = 1, 2, \ldots, n$. The norm in $S(R_+, R^n)$ is simply defined by $|x|_{S(R_+, R^n)} = |\,\|x(t)\|\,|_{S(R_+, R)}$.

Using the weighting scheme due to Gollwitzer, we can define now the Banach function space $S_g(R_+, R^n)$, starting from $S(R_+, R^n)$. The subscript g above stands again for a measurable map from R_+ into $L(R^n, R^n)$. As shown by Gollwitzer [27], $g_{-1}(t)$ is also measurable.

The space $S_g(R_+, R^n)$ is now defined as follows: $x : R_+ \to R^n$ belongs to S_g iff it is measurable, $P_{g(t)}x(t) = x(t)$ a.e. on R_+, and $g_{-1}(t)\, x(t) \in S(R_+, R^n)$. The norm of S_g is given by

$$(19) \qquad |x|_{S_g} = |g_{-1}(t)\, x(t)|_S\,.$$

The following lemma [31] clarifies the structure of S_g, as a normed space.

Lemma. *The space $S_g(R_+, R^n)$ is a Banach function space.*

Proof. First, let us notice that $|x|_{S_g}$ is a norm on S_g. Only fact to be checked is that $|x|_{S_g} = 0$ implies $x(t) = 0$ a.e. on R_+. Indeed, if $x \in S_g$ is such that $|x|_{S_g} = 0$, then $g_{-1}(t)\, x(t) = 0$ a.e. on R_+, from which we obtain $g(t) g_{-1}(t) x(t) = 0$ a.e. on R_+. But according to the definition $g(t) g_{-1}(t)\, x(t) = x(t)$ a.e. on R_+. Hence, $|x|_{S_g}$ is a norm on S_g.

Assume now that $\{x_n\} \subset S_g$ is a Cauchy sequence. This implies that $\{y_n\}$, $y_n(t) = g_{-1}(t)\, x_n(t)$, $n \geqslant 1$, is a Cauchy sequence in S. Therefore, there exists $y \in S$, such that $|y_n - y|_S \to 0$ as $n \to \infty$. Consider now the map $x(t) = g(t)\, y(t)$, a,e, on R_+. Since $y_h(t) \in N^+_{g(t)}$ a.e., one obtains $y(t) \in N^+_{g(t)}$. a.e. on R_+. This means $P_{g(t)}\, x(t) = x(t)$ a.e. on R_+. Moreover, since $|x_n - x|_{S_g} = |y_n - y|_S \to 0$ as $n \to \infty$, one concludes that S_g is a Banach space.

Can the result of Theorem 2 be generalized to the spaces S_g?

A procedure of defining Banach function spaces, more general than that leading to the spaces C_g or L^p_g, with g scalar, is due to A. P. Calderon [11]. M. Milman [30], [31], has brought interesting contributions to the theory of Calderon spaces, and related them to the admissibility theory with respect to integral operators, the tensor products, interpolation spaces, and other concepts. We think these generalizations are somewhat overreaching the aim of this presentation. Therefore, we send the interested reader to the work of M. Milman quoted above.

It should be noted that weighted spaces have been dealt with by many authors, especially in the theory of partial differential equations. The purpose was usually different than in Bielecki's approach (for instance, describing behavior of functions at the boundary).

In [9], P. J. Bushell defines a class of spaces which he also denotes by $C_g = C_g((0,1], R)$, in connection with the investigation of some nonlinear integral equations of the form

$$x(t) = f(t) + \int_0^t k(t, s)\, \{x(s)\}^p ds\,,$$

or

$$x(t) = f(t) + \int_0^1 k(t, s)\, \{x(s)\}^p ds\,.$$

The "singularity" of the equations is not at infinity anymore, but at the origin ($t = 0$). The norm is given by

(20) $$|x|_g = \sup\left\{ |x(t)| / g(t), t \in (0, 1] \right\},$$

and it is assumed that the right hand side in (20) is finite.

It is obvious that these spaces are isomorphic and isometric to the spaces $C_g(R_+, R)$ considered above (the case of positive $g(t)$ on $(0, 1]$).

In concluding this section, we shall briefly deal with another generalization of the spaces C_g, due to M. Turinici [40], [41], who used it in connection with some integro-differential equations. We assume again that g is the scalar weight functions used in [15], but instead of a Banach function space one obtains a generalized Banach space (in the sense of generalized spaces of Luxemburg, when the distance between two elements can be also $+\infty$).

The space $(C, |\cdot|_g) = \tilde{C}_g$ will consist of all continuous maps from R_+ into R^n, where the generalized norm is given by

(21) $$|x|_g = \begin{cases} \inf\left\{ \lambda \mid \lambda \in R_+,\ \|x(t)\| \leqslant \lambda g(t),\ t \in R_+ \right\}, \\ \qquad\qquad \text{when } \left\{ \lambda \mid \lambda \in R_+,\ \|x(t)\| \leqslant \lambda g(t) \right\} \neq \phi, \\ +\infty, \text{ otherwise.} \end{cases}$$

Endowed with the generalized norm (21), $\tilde{C}_g$ becomes a complete generalized metric space. C_g, as defined above, is isometrically imbedded in $\tilde{C}_g$.

Admissibility. The concept of admissibility with respect to an integral operator can be briefly described as follows: given two function spaces, say $X = X(R_+, R^n)$ and $Y = Y(R_+, R^n)$, and an integral operator

(22) $$(Kx)(t) = \int_0^\infty k(t, s)\, x(s)\, ds,$$

with $k : \Delta \to \mathrm{L}(R^n, R^n)$ measurable, find conditions on k such that

(23) $$KX \subset Y.$$

Sometimes, it appears necessary to add other properties to the inclusion (23). Usually, when X and Y stand for some Banach function space, the inclusion (23) implies the continuity on the operator K defined by (22). This fact is derived from the closed graph theorem.

Of course, the literature on this theme is very rich, and we confine our interest to the cases dealing with the classes of spaces considered in the preceding section.

The term admissibility, in connection with the above scheme, has been coined by J. L. Massera and J. J. Schaffer, in connection with the ordinary differential equations.

In regard to the theory of integral equations, this term has been used by the author in [15], [16]. A good deal of results pertaining to this concept have been reviewed in the survey paper [43] by L. B. Tzalyuk, where a special section is dedicated to "Admissibility". Another reference paper, mainly concerned with Soviet contributions to this topic, is the monograph of V. A. Tychkevich [42]. The literature prior to 1970 is reviewed in author's book [21].

It should be stressed that the admissibility with respect to Volterra integral operators is not necessarily simpler than in case of Fredholm operators of the form (22). Under more assumptions (than measurability) on the kernel $k(t, s)$, it happens sometimes to have some of the conditions automatically satisfied by the Volterra kernels. We shall point out such situations in the sequel, whenever they arise.

Let us give now a result due to M. Milman [31], which constitutes one of the most general results in admissibility theory with respect to integral operators.

First, a few definitions are necessary in order to formulate the result.

The Banach function space $X(R_+, R)$, as defined in the preceding section, is said to have an absolutely continuous norm, if and only if the following property holds true: for every $f \in X$, and every sequence $\{A_n\}$ of measurable sets on R_+, such that $A_n \downarrow \emptyset$, one has $|f\chi_{A_n}|_X \to 0$ as $n \to \infty$.

The associate space X' to the Banach function space X is defined by

$$(24) \qquad X' = \left\{ y \,|\, y \text{ measurable, and } \sup \int_{R_+} |x(t)\,y(t)|\,dt < \infty \text{ for } |x|_X \leq 1 \right\}.$$

The norm in X' is the sup appearing in (24).

Let $g : R_+ \to \mathbf{L}(R^n, R^n)$ be a measurable weight function, and assume that $h : R_+ \to \mathbf{L}(R^n, R^n)$ is continuous. We shall be concerned with the admissibility of the pair of spaces $X_g(R_+, R^n)$, $C_h(R_+, R^n)$, with respect to the operator K defined by (22). In other words, we look for conditions assuring the inclusion

$$(25) \qquad K X_g \subset C_h .$$

Motivated by the needs of the theory of integral equations, it is useful to impose one more condition on the operator and spaces involved. Namely, we shall be interested in such operators that take every bounded set of X_g, into a set of C_h which is equicontinuous at every $t_0 \in R_+$. In such case, we shall say that the triplet (X_g, C_h, K) defines a strongly admissible (or strongly stable) systems.

Theorem 3. *A set of necessary and sufficient conditions for the strong admissibility of the system (X_g, C_h, K) consists of the following:*

a) $P_{h(t)}\left\{ \int_0^\infty k(t,s)\,x(s)\,ds \right\} = \int_0^\infty k(t,s)\,x(s)\,ds, \ \forall t \in R_+, \ x \in X_g$;

b) $|h_{-1}(t)\,k(t,\cdot)\,g(\cdot)|_{X'} \in L^\infty$;

c) $\lim\limits_{t \to t_0} |h(\cdot)\left\{k(t,\cdot) - k(t_0,\cdot)\right\}|_{X'} = 0, \ \forall t_0 \in R_+$.

Remark. The condition a) can be written equivalently as $P_{h(t)}\, k(t,s)g(s) = k(t,s)g(s)$, $(t,s) \in \Delta$, due to the fact $x \in X_g$ is arbitrary.

The proof of Theorem 3 is given in [31], for the particular case $g = I$. It can be easily carried out to an arbitrary weight function $g(t)$.

Theorem 3 contains as special cases most of the admissibility results obtained earlier, under the hypothesis that the range of K is in a Banach space of continuous functions.

For instance with $X = L^p$, $p > 1$, and $g : R_+ \to \mathrm{L}(R^n, R^n)$ an arbitrary measurable map, the result has been obtained by Gr. Dotseth [25]. In this case, $X' = L^q$, where $p^{-1} + q^{-1} = 1$, and conditions b) and c) of Theorem 3 take an integral form.

We shall now state another admissibility result, due to Gr. Dotseth [25], which is not a special case of Theorem 3 above.

Theorem 4. *Let $g, h : R_+ \to \mathrm{L}(R^n, R^n)$ be two measurable maps, and assume the kernel $k : R_+ \times R_+ \to \mathrm{L}(R^n, R^n)$ is also measurable. Then the pair (L_g, L_h) is admissible with respect to the operator K given by (2), if and only if*

$$(26)\qquad P_{h(t)}\left\{\int_0^\infty k(t,s)\,x(s)\,ds\right\} = \int_0^\infty k(t,s)\,x(s)\,ds, \ \text{a.e. on } R_+,\ \forall x \in L_g^\infty,$$

and, there exists $M > 0$ such that

$$(27)\qquad \int_0 \| h_{-1}(t)\,k(t,s)\,g(s) \|\, ds \leqslant M, \ \text{a.e. on } R_+.$$

The proof of this theorem can be carried out on the same lines as its scalar counterpart (i.e., g and h are scalar weight functions) [21].

A final results on admissibility with respect to integral operators, is due to H. Gollwitzer [27], and deals with Volterra operators and C_g spaces.

Theorem 5. *Let $g, h : R_+ \to \mathrm{L}(R^n, R^n)$ be two continuous weight functions, and consider the Volterra operator*

$$(28)\qquad (Kx)(t) = \int_0^t k(t,s)\,x(s)\,ds,$$

where $k : \Delta \to \mathrm{L}(R^n, R^n)$ is continuous. The pair (C_g, C_h) is admissible with respect to the operator K given by (28), if and only if

$$(29)\qquad P_{h(t)}\left\{\int_0^t k(t,s)\,x(s)\,ds\right\} = \int_0^t k(t,s)\,x(s)\,ds,\ 0 \leqslant s \leqslant t,\ \forall x \in C_g,$$

and

$$(30)\qquad \int_0^t \| h_{-1}(t)\,k(t,s) \|\, ds \leqslant M,\ t \in R_+,$$

for some positive constant $M = M(g, h, k)$.

The proof of Theorem 5 is provided in [27]. The scalar case is dealt with in [15] or [21].

If we compare the Theorems 3 and 5, we notice that condition c) of Theorem 3 does not have a counterpart in Theorem 5. Of course, this particularity is easily explained by the fact that condition c) in Theorem 3 is imposed to obtain continuity, while in Theorem 5 the continuity of $(Kx)(t)$, $x \in C_g$, is secured by the continuity of k, and the fact that K in (28) is of Volterra type.

It can also be seen that the triplet (C_g, C_h, K) with K given by (28), is strongly stable (or admissible in the sense of the definition formulated in this section, if conditions of Theorem 5 hold true.

The admissibility results quoted above involve only the function spaces defined by means of the weighting scheme due to H. E. Gollwitzer [27]. It is, of course, interesting to quote results in which different kinds of function spaces are involved. An interesting case in this respect has been dealt with by G. Bantas [3], and then generalized by Gr. Dotseth [25]. We shall state this result here, considering only Volterra type integral operators.

First, we shall denote by $C_\ell = C_\ell(R_+, R^n)$ the Banach space of continuous maps from R_+ into R^n such that $\lim x(t) = x(\infty) \in R^n$ as $t \to \infty$. The norm will be the supremum norm, which induces the uniform convergence on the whole R_+.

The Volterra kernel $k(t, s)$ is assumed to be continuous on Δ, and moreover

$$(31) \quad \lim_{t \to \infty} k(t, s) = k(s), s \in R_+,$$

uniformly on any compact subset of R_+.

Theorem 6. *Let $g : R_+ \to \mathrm{L}(R^n, R^n)$ be a continuous map, and assume $k(t, s)$ is a continuous Volterra kernel for which (31) holds true.*

The pair (C_g, C_ℓ) is admissible with respect to the operator K, given by (28), if and only if

$$(32) \quad \int_0^\infty \| k(t, s) g(s) \| \, ds < +\infty,$$

and

$$(33) \quad \lim_{t \to \infty} \int_0^t \| k(t, s) g(s) \| \, ds = \int_0^\infty \| k(s) g(s) \| \, ds.$$

It would be interesting to extend this result to the case of integral operators of the form (22).

Finally, there are several ways to investigate admissibility results, in which integral operators/equations are involved. In connection with the C_g spaces, a concept of admissibility with respect to a Volterra integral equation has been dealt with by the author in [17], [21]. Since next section of this survey is dedicated to the existence of solutions of integral equations, we shall discuss this kind of admissibility there.

Other admissibility results can be found in the papers by C. Avramescu, G. Bantas, D. Petrovanu, quoted in the reference list.

Existence Results for Integral Equations. We shall use, in this section, the admissibility theory of pairs of function spaces, in order to find existence (and uniqueness) results for some classes of nonlinear integral equations. We shall consider equations of Hammerstein type, such as

(34) $$x(t) = p(t) + \int_0^\infty k(t, s) f(s; x)\, ds, \; t \in R_+,$$

where $f(t; x) = (fx)(t)$ is an operator acting between convenient function spaces. We shall also consider Volterra type equations of the form

(35) $$x(t) = p(t) + \int_0^t p(t, s) f(s; x)\, ds, \; t \in R_+,$$

as well as perturbed equations of some linear Volterra equations. For instance if one starts with the linear equation

(36) $$x(t) = p(t) + \int_0^t k(t, s)\, x(s)\, ds, \; t \in R_+,$$

one can attach to it the perturbed equation

(37) $$x(t) = p(t; x) + \int_0^t k(t, s)\, x(s)\, ds, \; t \in R_+,$$

which becomes for

(38) $$p(t; x) = p(t) + \int_0^t K(t, s, x(s))\, ds, \; t \in R_+,$$

(39) $$x(t) = p(t) + \int_0^t [k(t, s)\, x(s) + K(t, s, x(s))]\, ds, \; t \in R_+.$$

The operator $p(t; x) = (px)(t)$ has the same measuring as $f(t; x)$ in (35), and must satisfy further conditions to be specified below.

Of course, one can consider even more general equations, such as

(40) $$x(t) = p(t; x) + \int_0^\infty k(t, s) f(s; x)\, ds, \; t \in R_+,$$

since the admissibility conditions have been usually formulated for Fredholm type operators (see the preceding section), and fixed point methods apply the same way.

Another class of equations to which the admissibility results have been applied by several authors [16], [39], [42] is the clas of Volterra integro-differential equations of the form

(41) $$\dot{x}(t) = f(t, x_t), \; t \in R_+,$$

where $x_t(s) = x(s)$, for $0 \leqslant s \leqslant t$. In particular, (41) can be of the form

(42) $$\dot{x}(t) = f\left(t, x(t), \int_0^t K(t, s, x(s))\, ds\right), \; t \in R_+,$$

or

$$(43)\quad \dot{x}(t)=f(t,\int_0^t x(s)\,d_sG(t,s)),\ t\in R_+,$$

and most of the times it appears as special case of (41) or (42).

We also notice the fact that only C_g or related spaces with scalar weight function have been considered in the literature on this subject, excepting [25], [26], and [27].

In the remaining part of this section we shall give, with at least a sketch of the proof, a few existence results for integral equations. These results will make use of a fixed point theorem, and will involve some admissibility results. As proceeded above, we shall restrict our considerations to the case of function spaces with a weighted norm, which will allow us to use the ideas developed by several authors in connection with Bielecki's procedure in proving existence theorems.

We shall give first a result which is based on Theorem 3 above. The conditions of that theorem will be preserved in stating the existence result.

Theorem 7. *Consider a triplet* (X_g, C_h, K), *subject to the same conditions as in Theorem 3, and assume further that in equation* (34) *one has* $p\in C_h$, *while* $f: C_h\to X_g$ *is such that*

$$(44)\quad |fx-fy|_{X_g}\leq \lambda\,|x-y|_{C_h},$$

for any $x,y\in C_h$, *with* $\lambda>0$ *a constant.*

Then, there exists a unique solution $x\in C_h$ *for equation* (34), *provided*

$$(45)\quad \lambda<(\,|h_{-1}(t)\,k(t,\cdot)\,g(\cdot)|_{X'})^{-1}.$$

The proof of Theorem 7 follows from Banach fixed point theorem, applied to the operator

$$(46)\quad (Tx)(t)=p(t)+\int_0^\infty k(t,s)f(s;x)\,ds,$$

in the underlying space C_h.

Since strong stability of the triplet (X_g, C_h, K) implies local equicontinuity for any set of functions $KB\subset C_h$, with $B\subset X_g$ bounded, one can easily formulate and existence result for (34), based on the application of Schauder-Tykhonov fixed point theorem (see [21] for examples of this kind).

Another result on existence and uniqueness for equation (34) has been obtained by Gr. Dotseth [25].

Theorem 8. *Consider the triplet* $(L_g^\infty, L_h^\infty, K)$ *under conditions of Theorem 4, and assume that in equation* (34) $p(t)\in L_h^\infty$, *while* $f: L_h^\infty\to L_g^\infty$ *is an operator satysfying*

$$(47)\quad |fx-fy|_{L_g^\infty}\leq \lambda\,|x-y|_{L_h^\infty},$$

for any $x, y \in L^{\infty}_{h}$, and for some $\lambda > 0$. Then equation (34) *has a unique solution $x \in L^{\infty}_{h}$, provided $\lambda < M^{-1}$.*

The proof of Theorem 8 follows easily from the fact that the operator T, defined by (46), is a contraction in the space L^{∞}_{h}. The main difference, with respect to the preceding theorem, consists in the fact that the existence of a measurable solution is stated.

A similar result to Theorem 8 can be obtained from Theorem 5 above, under continuity assumptions. The triplet (C_g, C_h, K), with K given by (28), is strongly stable if conditions (29) and (30) of Theorem 5 are satisfied (H. Gollwitzer, [27]). One obtains the existence and uniqueness of a continuous solution for the Volterra equation (35). See [25] for more details.

Next result is due to Gr. Dotseth [25], and deals with the existence of solutions possessing a finite limit at infinity. The admissibility result taken as a starting point is given in Theorem 6 above.

Theorem 9. *Consider the equation* (35), *and assume that the triplet (C_g, C_ℓ, K), with K given by* (28), *satisfies the assumptions of Theorem 6. If, moreover, $p(t) \in C_\ell$, and $f : C_\ell \to C_g$ verifies a Lipschitz condition with sufficiently small constant, then there exists a unique solution $x \in C_\ell$ of the equation* (35).

Somewhat different results on the existence and uniqueness of solutions to integral equations can be obtained using the concept of admissibility with respect to an integral equation (instead of an integral operator, as above).

The pair of spaces (B, D) will be called admissible with respect to the equation (36), if for any $p \in B$, the solution x of (36) belongs to $D : x \in D$.

It is well known that for any continuous Volterra kernel $k(t, s)$, there exists a resolvent kernel $r(t, s)$, continuous on Δ, such that the solution of (36) is given by

$$(48) \qquad x(t) = p(t) + \int_0^t r(t, s)\, p(s)\, ds, \quad t \geqslant 0.$$

Hence, the admissibility with to a Volterra equation can be reduced, roughly speaking, to the admissibility with respect to the integral operator generated by the corresponding resolvent kernel. Results in this regard can be found in [17], [21], [27].

We shall include here a result of this kind, which is somewhat more general than the one given in [17] for the case of scalar weight functions.

Theorem 10. *The pair (C_g, C_h) is admissible with respect to equation* (36), *if and only if the following conditions are satisfied:*

a) $P_h(t)\, g(t) = g(t), \ \forall\, t \in R_+$;

b) $P_h(t)\, r(t, s)\, g(s) = r(t, s)\, g(s), \ \forall\, (t, s) \in \Delta$;

c) *there exists a positive constant A, such that*

$$(49)\quad \| h_{-1}(t)g(t)\| + \int_0^t \| h_{-1}(t)r(t,s)g(s)\| \, ds \leq A, \quad t \in R_+.$$

The proof of Theorem 10, with g and h matrix valued functions, can be found in [27]. The scalar case is dealt with in [17] and [21].

Theorem 10 can be used to obtain further existence results for nonlinear Volterra equations.

For instance, if one considers the equation (37), which appears as a perturbed equation for (36), it can be rewritten as

$$(50)\quad x(t) = p(t;x) + \int_0^t r(t,s)p(s;x)\, ds, \quad t \geq 0.$$

If $p : C_h \to C_g$ satisfies a Lipschitz condition with sufficientlly small constant, then (50) has a unique solution $x \in C_h$. In particular, equation (39) can be dealt with within this framework.

We will conclude the paper with a reference to some recent results of O. Staffans [37], related to the admissibility properties of the resolvent kernel. As one can see from Theorem 10, the resolvent kernel is involved in condition (49). Conditions a) and b) are automatically satisfied when g and h are scalar weight functions.

Of course, it would be desirable to substitute to (49) something involving the kernel $k(t, s)$ of the equation (36), instead of $r(t, s)$. While $r(t, s)$ is known to exist, it is not an easy matter to determine if effectively. Therefore, it is not easy to check the validity of (49). In [37], the author brings some interesting contributions which allow to infer (49), based on properties directly formulated on $k(t, s)$ and the weighting (scalar) functions.

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STRESZCZENIE

W 1956 r. A. Bielecki podał nową metodę otrzymywania globalnych twierdzeń egzystencjalnych dla równań funkcyjnych, polegającą na wprowadzeniu nowej normy, względem której operator związany z danym równaniem staje się kontrakcją. Metoda Bieleckiego była stosowana dość szeroko przez wielu autorów do zagadnień istnienia i jedyności rozwiązań równań różniczkowych, całkowych, całkowo-różniczkowych etc. tak, że szereg autorów nie poczuwa się do obowiązku cytowania jej twórcy.

Praca niniejsza stanowi przegląd rezultatów otrzymanych tą metodą w ciągu ostatnich 20 lat.

РЕЗЮМЕ

В 1956 году А. Белецки представил новый метод устанавливания глобальных, экзистенциальных теорем для функциональных уравнений состоящий в определении новой нормы, относительно которой рассматриваемый оператор становится контракцией. В дальнейшем, метод Белецкого был использован многими авторами в изучении вопросов существования и однозначности решений разных типов уравнений, например: дифференциальных, интегральных, интегро-дифференциальных и других. И так, по ходу дела, ряд авторов не чувствует уже себя обязанными цитировать создателя этого метода. В данной статье приведен обзор результатов полученых этим методом за последние 20 лет.