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The Grunsky Coefficients of Meromorphic Starlike and Convex Functions

Współczynniki Grunsky'ego funkcji meromorficznych gwiaździstych i wypukłych

Коэффициенты Грунского мероморфных, звёздных и выпуклых функций

1. Introduction

Let Σ^* be the class of functions

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$$

analytic and starlike in $|z| > 1$, and Σ_0^* the subset of Σ^* corresponding to the particular case $a_0 = 0$. Let Σ^k be the class of functions of the form (1) analytic and convex in $|z| > 1$; thus $f \in \Sigma^k$ if and only if $zf' \in \Sigma_0^*$ [2; p. 47].

The Grunsky coefficients $a_{m,n}$ of meromorphic univalent functions f are defined by the relation

$$\log \frac{f(\zeta) - f(z)}{\zeta - z} = -\sum_{m,n=1}^{\infty} a_{m,n} \zeta^{-m} z^{-n};$$

the Grunsky inequalities [2; Chap. 3] show at once that

$$|a_{m,n}| \leqslant (mn)^{-\frac{1}{2}}. \tag{2}$$

A well-known result of Clunie [2; p. 48, Theorem 2.10] shows that, if $f \in \Sigma^*$, then

$$|a_n| = |a_{1,n}| \leqslant 2(n+1)^{-1}. \tag{3}$$

Numerical computations by Miss H. Bökemeier and C. Pommerenke [3] led to their conjecture that

$$|a_{m,n}| \leqslant 2(m+n)^{-1} (f \in \Sigma^*). \tag{4}$$

For $m = n$, (2) and (4) are the same; and, for the function $f(z) = z + z^{-1}$, $a_{n,n} = n^{-1}$ for all n.

In §§ 2-3 we prove a special case of (4).

Theorem: *Let $f \in \Sigma_0^*$ be of the form* (1). *Then*

(5) $$|a_{2,3}| \leqslant 2/5, \text{ and}$$

(6) $$|a_{2,4}| \leqslant 2/6.$$

Our methods do not seem to give (4) either for $a_{2,3}$ for Σ^* or for other $a_{m,n}$ for Σ_0^*.

In § 4 we give some examples which show that the inequality

$$|a_{m,n}| \leqslant 2/(m+n)(m+n-1) \; (m \neq n, f \in \Sigma^k).$$

somewhat analogous to (4), is not true in general; it seems difficult to see what the sharp upper bounds for $a_{m,n}$ might be in this case.

2. Proof of (5)

Let P_0 denote the class of functions

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^{-n}$$

analytic and of positive real part in $|z| > 1$. Since here $f \in \Sigma_0^*$,

(7) $$zf'(z)/f(z) = p(z)$$

for some $p \in P_0$. From (7) it follows that $-2a_1 = p_2$, $-3a_2 = p_3$, $-4a_3 = p_4 - \frac{1}{2}p_2^2$, $-5a_4 = p_5 - \frac{5}{6}p_2p_3$ and $-6a_5 = p_6 - \frac{1}{4}p_2p_4 - \frac{1}{3}p_3^2 + \frac{1}{8}p_2^3$. Then (3) gives

(8) $$|p_5 - \tfrac{5}{6}p_2p_3| \leqslant 2.$$

Now, since $p \in P_0$, so does

$$1/p(z) = 1 - p_2z^{-2} - p_3z^{-3} + (-p_4 + p_2^2)z^{-4} + (-p_5 + 2p_2p_3)z^{-5} + \dots$$

and hence

(9) $$|p_5 - 2p_2p_3| \leqslant 2.$$

Then, since $a_{2,3} = a_4 + a_1a_2$ [2; p. 58],

$$|a_{2,3}| = \tfrac{1}{105}\,|6(p_5 - \tfrac{5}{6}p_2p_3) + 15(p_5 - 2p_2p_3)| \leqslant 2/5$$

using (8) and (9). It is easy to verify that equality holds in (5) only for functions of the form

$$f(z) = z(1 + \varepsilon z^{-5})^{2/5}, \; |\varepsilon| = 1.$$

3. Proof of (6)

From (7) and the identity $a_{2,4} = a_5 + a_1 a_3 + \frac{1}{2}a_2^2$ [2; p. 58] we find that

$$-6a_{2,4} = p_6 - \tfrac{2}{3}p_2p_4 + \tfrac{1}{2}p_2^3 - \tfrac{2}{3}p_3^2 \quad (p(z) \in P_0).$$

We now follow the ingenious method of Nehari and Netanyahu [1; especially p. 20] in their proof that $|a_5| \leqslant 1/3$. This means that we have to find functions

$$H(z) = 1 + \sum_{n=1}^{\infty} c_n z^{-n}, \quad h(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^{-n}$$

analytic and of positive real part in $|z| > 1$ that satisfy the following system of relations:

$$\begin{aligned} &\gamma_1 c_2 c_4 = 3, \ \gamma_2 c_2^3 = 4 \\ (10) \qquad &\gamma_1 c_3^2 = \tfrac{8}{3}, \ c_6 = 2 \\ &\gamma_1 = \tfrac{1}{2}(1 + \tfrac{1}{2}\beta_1), \ \gamma_2 = \tfrac{1}{4}(1 + \beta_1 + \tfrac{1}{2}\beta_2) \end{aligned}$$

Now (10) is satisfied by the following choices:
$h(z) = 1 + z^{-1}$ (and so $\beta_1 = 1$, $\beta_2 = 0$, $\gamma_1 = \frac{3}{4}$, $\gamma_2 = \frac{1}{2}$) and

$$H(z) = \lambda \frac{1+z^{-1}}{1-z^{-1}} + (1-\lambda)\frac{1-z^{-1}}{1+z^{-1}}$$

with $\lambda = \frac{1}{2} + \sqrt{2/9}$ (so that $c_2 = c_4 = c_6 = 2$, $c_3 = \sqrt{32/9}$). It follows from [1; equations (11) and (16b), with $n = 6$] that $|-6a_{2,4}| \leqslant 2$.

Note. A similar argument establishes (5), but we have preferred a simpler method here.

4. Grunsky coefficients for functions in Σ^k.

Note that $f \in \Sigma^k$ if and only if

$$(11) \qquad 1 + zf''(z)/f'(z) = p(z)$$

where $p \in P_0$ (as defined in § 2). From (11) we can check that

$$\begin{aligned} 20a_{2,3} &= p_5 + \tfrac{5}{6}p_2p_3, \\ 30a_{2,4} &= p_6 + \tfrac{1}{2}p_2p_4 + \tfrac{1}{12}p_3^2 - \tfrac{1}{2}p_2^3, \\ (12) \qquad 42a_{2,5} &= p_7 + \tfrac{7}{20}p_2p_5 - \tfrac{7}{8}p_2^2p_3, \text{ and} \\ 42a_{3,4} &= p_7 + \tfrac{7}{20}p_2p_5 + \tfrac{7}{12}p_3p_4 - \tfrac{27}{24}p_3^2p_3. \end{aligned}$$

Example A. Take

$$(13) \qquad 2(\cos\theta+\cos 2\theta)p(z) = \cos\theta\left[\frac{1+\omega z}{1-\omega z}+\frac{1+\overline{\omega}z}{1-\overline{\omega}z}\right]+\cos 2\theta\left[\frac{1+\omega^3 z}{1-\omega^3 z}+\frac{1+\overline{\omega}^3 z}{1-\overline{\omega}^3 z}\right],$$

where $\theta = \pi/5$, $\omega = e^{2i\theta}$. Then $p_1 = p_4 = 0$, $p_2 = p_3 = 2(\cos 2\theta-\cos\theta)$, $p_5 = 2$; and so $20a_{2,3} > 2$.

Example B. Take $p(z) = (1+z^3)/(1-z^3)$ so that $30a_{2,4} = 7/3 > 2$.

Example C. Take $p(z)$ as in (13), but with $\theta = \pi/7$, $\omega = e^{2i\theta}$. Then $p_1 = 0$ (by construction of $p(z)$), $p_2 = p_5 \neq 0$, $p_3 = p_4 < 0$; hence for this $f(z)$, $42a_{2,5} > 2$ and $42a_{3,4} > 2$.

REFERENCES

[1] Nehari, Z. and Netanyahu, E. *On the coefficients of meromorphic schlicht functions*, Proc. Amer. Math. Soc. 8 (1957), 15-23; MR 18, 648.

[2] Pommerenke, C. *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen 1975.

[3] Pommerenke, C., private communication.

STRESZCZENIE

W związku z hipotezą Pommerenke i Bökemeier, że wyrazy macierzy Grunsky'ego (a_{mn}) dla funkcji gwiaździstych spełniają nierówność

$$|a_{mn}| \leqslant 2(m+n)^{-1}$$

autor wykazuje tę nierówność w dwóch przypadkach specjalnych: $m = 2$, $n = 3$; $m = 2$, $n = 4$.

РЕЗЮМЕ

В связи с гипотезой Поммеренке и Бекемеер о том, что выражения матрицы Грунского (a_{mn}) для звёздной функции исполняют неравенство

$$|a_{mn}| \leqslant 2(m+n)^{-1}$$

автор доказывает это неравенство в двух специальных случаях: $m = 2$, $n = 3$; $m = 2$, $n = 4$.