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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

WOJCIECH ZYGMUNT

The Generic Property of Differential Equations with Compact Convex Valued Solutions

Własność generyczna równań różniczkowych, których rozwiązaniami są zbiory zwarte i wypukłe

Общее свойство уравнений в выпуклых компактных контингенциях

Introduction

Let R^n be the real n-dimensional Euclidean space with the usual norm $|\cdot|$. By C we denote the family of all nonempty compact convex subsets of R^n endowed with the Hausdorff metric d generated by the norm $|\cdot|$. It is known (see [4]) that (C, d) is a complete metric space. We shall denote by capital letters $X, Y, Z, \ldots$ elements of C. In C we introduce the usual algebraic operations:

addition: $X+Y = \{x+y\colon x\in X, y\in Y\}$;
multiplication by nonnegative scalars λ: $\lambda X = \{\lambda x\colon x\in X\}$.

The following properties hold (see [2]):

$$(*)\qquad \begin{array}{ll} X+\{\theta\} = \{\theta\}+X = X & 1\cdot X = X \\ X+(Y+Z) = (X+Y)+Z & \lambda(X+Y) = \lambda X+\lambda Y \\ X+Y = Y+X & \lambda(\mu X) = (\lambda\mu)X \\ & (\lambda+\mu)X = \lambda X+\mu X \end{array}$$

$$d(X+U, Y+V) \leqslant d(X, Y)+d(U, V)$$
$$d(X+U, Y+U) = d(X, Y)$$
$$d(\lambda X, \lambda Y) = \lambda d(X, Y)$$
$$d(\lambda X, \mu Y) \leqslant \beta d(X, Y)+|\lambda-\mu|\big(d(X, \{\theta\})+d(Y, \{\theta\})\big)$$

where $\beta = \max(\lambda, \mu)$, λ, μ are nonnegative real numbers, θ is the origin of R^n (i.e. $\theta = (0, 0, \ldots, 0)$) and $\{\theta\}$ denotes the set, whose unique element is θ.

Let $I = [0, 1]$ be an unit interval of the real line R. We shall say that a mapping $F\colon I\times C\to C$ is bounded if there is a positive number M such that $d(F(t, X), \{\theta\})\leqslant M$ for each $(t, X)\in I\times C$.
Let us denote by $\mathcal{F}$ the collection of all continuous bounded maps $F\colon I\times C\to C$. For $F, G\in\mathcal{F}$ we put

$$\operatorname{Dist}(F, G) = \sup\{d(F(t, X), G(t, X))\colon (t, X)\in I\times C\}.$$

Then the space $(\mathcal{F}, \operatorname{Dist})$ is a complete metric space.

In the present note we shall deal with the following differential problem of the type

$$(1)\qquad \begin{cases} \dot{X}(t) = F(t, X(t)), \\ X(0) = X_0, \quad X_0\in C \end{cases}$$

where $F\in\mathcal{F}$ and $\dot{X}$ denotes the Hukuhara derivative (see [5]) of the set valued function $X\colon I\to C$. By a solution of this problem we mean any continuous function X which satisfies (1) on I.

Using the Costello technique [3] we shall show that the set of mappings F for which the problem (1) has not an unique solution is a set of the first category in the space $(\mathcal{F}, \operatorname{Dist})$ (this property is called generic). We recall that a set is said to be of the first category if it is the countable union of nowhere dense and closed sets.

Main Theorem

Let $(1, F)$ denote the problem (1) with the right hand side F. Consider the set $\mathcal{X}$ defined by

$$\mathcal{X} = \{F\in\mathcal{F}\colon (1, F) \text{ has nonunique solutions}\}.$$

$\mathcal{X}$ consists of all mappings in $\mathcal{F}$ for which (1) has at least two solutions.

Theorem. *The set $\mathcal{X}$ is of the first category in $\mathcal{F}$.*

Before proving this theorem we shall state two lemmas that will be used in the proof.

Lemma 2. *Let $F\in\mathcal{F}$ and let $\delta > 0$ be given. Then there exists a locally Lipschitz mapping $G\in\mathcal{F}$ such that* $\operatorname{Dist}(F, G) < \delta$.

A mapping $G\colon I\times C\to C$ is called locally lipschitzean if for each point $p\in I\times C$ there is open neighbourhood $\mathcal{O}_p$ of p and L_p such that $d(G(t, X), G(s, Y))\leqslant L_p\varrho((t, X), (s, Y))$ for all $(t, X), (s, Y)\in\mathcal{O}_p$ where $\varrho((t, X), (s, Y)) = \max(|t-s|, d(X, Y))$. Since the proof of this lemma is essentially the same as the proof of Lemma 1 in [6] given by A. Lasota and J. Yorke, we only sketch it briefly here.

Define

$$N(\delta, (t, X)) = \{(s, Y) \in I \times C\colon\ \varrho((t, X), (s, Y)) < 1 \text{ and } d(F(t, X), F(s, Y)) < \delta\}.$$

There is a locally finite refinement $\{Q_\alpha\}_{\alpha \in \mathscr{A}}$ of $\{N(\delta/2, (t, X))\colon (t, X) \in I \times C\}$ where each Q_α is nonempty and open.
For $\alpha \in A$ we define $\mu_\alpha\colon I \times C \to [0, \infty)$, $p_\alpha\colon I \times C \to I$ as follows

$$\mu_\alpha(t, X) = \begin{cases} 0 & \text{if } (t, X) \notin Q_\alpha \\ \inf\limits_{(s, Y) \in \partial Q_\alpha} \varrho((t, X), (s, Y)) & \text{if } (t, X) \in Q_\alpha \\ \partial Q_\alpha \text{ denotes the boundary of } Q_\alpha), & \end{cases}$$

$$p_\alpha(t, X) = \mu_\alpha(t, X)\Big(\sum_{\beta \in \mathscr{A}} \mu_\beta(t, X)\Big)^{-1}.$$

Then each p_α is locally lipschitzean.
Let $\{(t_\alpha, X_\alpha)\}$ be a set of points such that $(t_\alpha, X_\alpha) \in Q_\alpha$ for all α.
Define now $G\colon I \times C \to I$ by

$$G(t, X) = \sum_{\alpha \in \mathscr{A}} p_\alpha(t, X) \cdot F(t_\alpha, X_\alpha).$$

It is easy to verify that G is well defined, because $\{Q_\alpha\}$ is locally finite, and locally lipschitzean in view of formulas $(*)$.
For each $(t, X) \in I \times C$ we have

$$d(F(t, X), G(t, X)) = d\Big(\sum_{\alpha \in \mathscr{A}} p_\alpha(t, X) F(t, X),\ \sum_{\alpha \in \mathscr{A}} p_\alpha(t, X) F(t_\alpha, X_\alpha)\Big)$$

$$\leqslant \sum_{\alpha \in \mathscr{A}} p_\alpha((t, X) d(F(t, X), F(t_\alpha, X_\alpha)) \leqslant \sum_{\alpha \in \mathscr{A}} (p_\alpha(t, X)) \cdot \delta = \delta.$$

Hence it follows that $\mathrm{Dist}(F, G) < \delta$.

Lemma 2. *If $F \in \mathscr{F}$ is locally lipschitzean, then the problem* (1) *has exactly one solution.*
To prove this lemma, let us recall the fact the space (C, d) may be embedded as a closed positive convex cone of a Banach space $(\mathscr{B}, \|\cdot\|)$ (see [1]) in such way that the embedding J is an isometric isomorphism, i.e.

$$J(\lambda X + \mu Y) = \lambda J(X) + \mu J(Y) \qquad \lambda, \mu \geqslant 0$$

and

$$\|J(X) - J(Y)\| = d(X, Y), \quad \text{where } X, Y \in C.$$

From that, using the embedding of equation (1) in the Banach space we obtain, in our hypotheses, the existence and uniqueness of the solution $X\colon I \to C$.

Proof of the theorem. By De Blasi and Iervolino Theorem (see [2]) each $(1, F)$ has at least one solution. Define

$$\gamma(F) = \sup\{\sup_{t\in I} d(X_1(t), X_2(t)) \colon X_1, X_2 \text{ solutions of } (1, F)\}$$

and

$$T_n = \left\{F \in \mathcal{F} \colon \gamma(F) \geqslant \frac{1}{n}\right\}, \quad n = 1, 2, \ldots$$

Then we have $\mathcal{X} = \bigcup_{n=1}^{\infty} T_n$.

It is easy to verify that each T_n is a closed set in $\mathcal{F}$. Now we shall show that each T_n is also a nowhere dense set. Let $F \in T_n$ be arbitrary. Fix $\varepsilon > 0$ arbitrarily and consider a neighborhood $\mathcal{O}_\varepsilon(F)$ of F of radius ε. By lemma 1 there is a locally Lipschitz mapping G_ε such that $G_\varepsilon \in \mathcal{O}_\varepsilon(F)$. This implies that each neighbourhood of F contains a mapping locally lipschitzean. In view of lemma 2, $(1, G_\varepsilon)$ has the unique solution. Hence no point of T_n has a neighbourhood contained in T_n, that is, T_n is a nowhere dense set in $\mathcal{F}$.

So $\mathcal{X}$ is a set of the first category in $\mathcal{F}$.

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STRESZCZENIE

W pracy udowodniono, że zbiór tych przekształceń F, dla których problem

(1) $$\dot{X}(t) = F(t, X(t)),$$

gdzie $\dot{X}$ oznacza pochodną w sensie Hukuhary funkcji wieloznacznej $X: I \to C$ ma przynajmniej dwa rozwiązania, jest zbiorem pierwszej kategorii w przestrzeni $(\mathscr{F}, \mathrm{Dist})$.

РЕЗЮМЕ

В работе доказано, что множество тех отображений F, для которых проблема

$$\begin{cases} \dot{X}(t) = F(t, X(t)) \\ X(0) = X_0, X_0 \in C, \end{cases}$$

где $\dot{X}$ обозначает производную по Фукухару многозначной функции $X: I \to C$ имеет неединственное решение, является множеством первой категории в пространстве $(\mathscr{F}, \mathrm{Dist})$.