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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

JAN STANKIEWICZ

The Influence of Coefficients on some Properties of Regular Functions

Wpływ współczynników na pewne własności funkcji regularnych

Влияние коэффициентов на некоторые свойства регулярных функций

1. Let T be a fixed subclass of the functions of the form

(1) $$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

which are regular in the unit disc $K_1 (K_r = \{z : |z| < r\})$ and having the following property

(2) $$f(z) \in T \Rightarrow \underset{0<r\leqslant 1}{\forall} f(rz) \text{ or } \frac{1}{r} f(rz) \text{ belongs to } T.$$

We say that the function $f(z)$ belongs to T in K_r or that $f(z)$ is the function of the class T in K_r if the function $f(rz)$ or $f(rz)/r$ belongs to T.

For the given function $f(z) \in T$ we change the coefficients a_n into $\tilde{a}_n$ and form a new function

$$\tilde{f}(z) = \tilde{a}_0 + \tilde{a}_1 z + \tilde{a}_2 z^2 + \ldots .$$

This new function $\tilde{f}(z)$ in general does not belong to T in K_1, but it may belong to T in a smaller disc K_r, $0 < r < 1$.

In this paper we want to find the maximum of these radii r for which $f(z) \in T \Rightarrow \tilde{f}(z) \in T$ in K_r when T is the class of Caratheodory functions with positive real part in K_1 or when T is the class of functions with bounded rotation.

This problem when $\tilde{a}_n = a_n e^{i\theta_n}$ for some n was investigated by S.A. Kasyanyuk and G.I. Tkačuk [1]. Q.I. Rahman [3] investigated the problem of the influence of coefficients on the zeros of polynomials.

2. Let $\{n_k\}$ be a finite or infinite subsequence of the sequence of natural numbers and $\{\varepsilon_{n_k}\}$ be a corresponding sequence of complex numbers and

let $f(z)$ of the form (1) belong to T. Let us put

$$\tilde{a}_m = \begin{cases} a_m & \text{if } m \neq n_k,\ k = 1, 2, \ldots \\ a_{n_k} + \varepsilon_{n_k} a_{n_k} & \text{if } m = n_k \text{ for some } k, \end{cases} \tag{3}$$

and

$$\tilde{f}(z) = \sum_{m=0}^{\infty} \tilde{a}_m z^m = f(z) + \sum_k \varepsilon_{n_k} a_{n_k} z^{n_k}. \tag{4}$$

Definition 1. For fixed $\{n_k\}$, $\{\varepsilon_{n_k}\}$ and $f(z) \in T$ let us denote by

$$R_1 = R_1(f, \{\varepsilon_{n_k}\}, T)$$

the radius of the largest disc K_r such that $f(rz)$ or $f(rz)/r$ belongs to T in K_r.

Definition 2. For fixed $\{n_k\}$, ϱ and $f(z) \in T$ we put

$$R_2 = R_2(f, \{n_k\}, \varrho, T) = \inf_{|\varepsilon_{n_k}| \leqslant \varrho} R_1(f, \{\varepsilon_{n_k}\}, T).$$

Definition 3. For fixed $\{n_k\}$ and $\{\varepsilon_{n_k}\}$ we put $R_3 = R_3(\{\varepsilon_{n_k}\}, T) = \inf\limits_{f \in T} R_1(f, \{\varepsilon_{n_k}\}, T)$.

Definition 4. For fixed $\{n_k\}$ and ϱ we put $R_4 = R_4(\{n_k\}, \varrho, T) = \inf\limits_{f \in T, |\varepsilon_{n_k}| \leqslant \varrho} R_1(f, \varepsilon_{n_k}\}, T) = \inf\limits_{f \in T} R_2(f, \{n_k\}, \varrho, T) = \inf\limits_{|\varepsilon_{n_k}| \leqslant \varrho} R_3(\{\varepsilon_{n_k}\}, T)$.

Remark 1. If we suppose that $|\varepsilon_{n_k}| \leqslant \varrho$ then

$$R_4 \leqslant \begin{Bmatrix} R_2 \\ R_3 \end{Bmatrix} \leqslant R_1.$$

3. Let P be a class of Carathéodory functions

$$f(z) = 1 + a_1 z + a_2 z^2 + \ldots \tag{5}$$

which are regular in the unit disc K_1 and satisfy the condition

$$\operatorname{Re} f(z) > 0 \text{ for } z \in K_1.$$

Remark 2. The function $f(z)$ belongs to P in K_r if and only if $f(z)$ has the form (5) and $\operatorname{Re} f(z) > 0$ in K_r.

In this part we shall find some estimations for R_k when $T = P$. Suppose for this and the next part that the sequences $\{n_k\}$ are finite or the series $\sum\limits_k |\varepsilon_{n_k} a_{n_k}| r^{n_k}$ are locally uniformly convergent in the interval $\langle 0, 1)$.

Theorem 1. *For fixed $\{n_k\}$, $\{\varepsilon_{n_k}\}$ and $f(z) \in P$ the number $R_1 = R_1(f, \{\varepsilon_{n_k}\}, P)$ is greater or equal to a positive root of the equation*

$$(1 + r) \sum_k |\varepsilon_{n_k} a_{n_k}| r^{n_k} + r - 1 = 0.$$

There exist such functions $f(z)\in P$ and such sequences $\{\varepsilon_{n_k}\}$ that the result is the best possible.

Theorem 2. *For fixed $\{n_k\}$, ϱ, and $f(z)\in P$ the number $R_2 = R_2(f, \{n_k\}, \varrho, P)$ is greater or equal to a positive root of the equation*

$$\varrho(1+r)\sum_k |a_{n_k}|r^{n_k}+r-1 = 0.$$

There exist such functions $f(z)\in P$ and such sequences $\{\varepsilon_{n_k}\}$, $|\varepsilon_{n_k}|\leqslant \varrho$ that $R_2 = R_1$ and the result is the best.

Theorem 3. *For fixed $\{n_k\}$ and $\{\varepsilon_{n_k}\}$ the number $R_3 = R_3(\{\varepsilon_{n_k}\}, P)$ is greater or equal to a positive root of the equation*

$$2(1+r)\sum_k |\varepsilon_{n_k}|r^{n_k}+r-1 = 0.$$

Theorem 4. *For fixed $\{n_k\}$ and ϱ the number $R_4 = R_4(\{n_k\}, \varrho, P)$ is equal to a positive root of the equation*

$$2\varrho(1+r)\sum_k r^{n_k}+r-1 = 0.$$

The extremal function is $f(z) = (1-z)/(1+z)$ and $\varepsilon_{n_k} = -\varrho(-1)^{n_k}$.

Now we give some remarks

Remark 3. Putting $\{n_k\} = \{2k\}$ we do not change the odd coefficients but only some even coefficients. Then R_4 is equal to a positive root of the equation

$$2\varrho(1+r)\sum_{k=1}^{\infty} r^{2k}+r-1 = 0$$

that is

$$(2\varrho-1)r^2+2r-1 = 0.$$

Thus

$$R_4 = \frac{1}{\sqrt{2\varrho}+1}.$$

Remark 4. Putting $\{n_k\} = \{2k-1\}_1^{\infty}$ we do not change the even coefficients but only some odd coefficients. Then R_4 is the positive root of the equation

$$r^2-2(1+\varrho)r+1 = 0.$$

Thus

$$R_4 = 1+\varrho-\sqrt{\varrho(\varrho+2)}.$$

Remark 5. If we put $\varepsilon_{n_k} = -1$ then $\tilde{a}_{n_k} = 0$, that is some coefficients of $\tilde{f}(z)$ vanish.

a) If $\{n_k\} = \{2k\}$, $\varepsilon_{n_k} = -1$ then $\tilde{f}(z) = 1 + \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1}$

and

$$R_3 = R_4 = \sqrt{2}-1.$$

The extremal function is $f(z) = (1-z)/(1+z)$.

b) If $\{n_k\} = \{2k-1\}$, $\varepsilon_{n_k} = 1$ then

$$R_3 = R_4 = 2-\sqrt{3}.$$

The extremal function is $f(z) = (1-z)/(1+z)$.

c) If $\{n_k\} = \{2k-1\}$, $\varepsilon_{n_k} = -1$ then

$$\tilde{f}(z) = 1 + \sum_{k=1}^{\infty} a_{2k} z^{2k} = \tfrac{1}{2}[f(z)+f(-z)] \in P,$$

that is $R_3 = 1$ is not equal to $R_4 = 2-\sqrt{3}$.

Remark 6. If we put

$$\{n_k\} = \{lk+m\} \quad l \geqslant 0, m \geqslant 1,$$

then R_4 is equal to a positive root of the equation

$$-r^{l+1} + r^l + 2\varrho(r^{m+1}+r^m) + r - 1 = 0, \text{ if } l \neq 0,$$

and

$$2\varrho(r^{m+1}+r^m) + r - 1 = 0 \text{ if } l = 0.$$

If $l = 0$ we change only one coefficient, $\tilde{a}_m = \varepsilon a_m$. For $l = 0$ and $m = 1$ we have

$$R_4 = \frac{2}{2\varrho+1+\sqrt{4\varrho^2+12\varrho+1}},$$

$$R_3 = \frac{2}{2|\varepsilon|+1+\sqrt{4|\varepsilon|^2+12|\varepsilon|+1}},$$

$$R_2 \geqslant \frac{2}{\varrho|a_1|+1+\sqrt{\varrho^2|a_1|^2+6\varrho|a_1|+1}},$$

$$R_1 \geqslant \frac{2}{|\varepsilon a_1|+1+\sqrt{|\varepsilon a_1|^2+6|\varepsilon a_1|+1}}.$$

Remark 7. For $k = 2$ or 4 we have

$$\lim_{\varrho\to 0} R_k = 1$$

and

$$\lim_{\varrho\to\infty} R_k = 0.$$

Proofs of Theorems 1–4. From (4) we have (for $|z| = r$)

$$\operatorname{Re}\tilde{f}(z) = \operatorname{Re} f(z) + \operatorname{Re}\sum_k a_{n_k}\varepsilon_{n_k} z^{n_k}$$

$$\geqslant \frac{1+r}{1+r} - \Big|\sum_k a_{nk}\varepsilon_{n_k} z^{n_k}\Big| \geqslant \frac{1-r}{1+r} - \sum_k |a_{n_k}\varepsilon_{n_k} z^{n_k}|,$$

that is

$$\operatorname{Re}\tilde{f}(z) \geqslant \frac{1-r}{1+r} - \sum_k |a_{n_k}\varepsilon_{n_k}| r^{n_k}.$$

From Remark 2 we have that $\tilde{f}(z)$ belongs to P in K_r if

$$\varphi(r) = \frac{1-r}{1+r} - \sum_k |a_{nk}\varepsilon_{nk}| r^{n_k} \geqslant 0.$$

This proves Theorem 1. If we take the infimum of $\varphi(r)$ with respect to all sequences $\{\varepsilon_{n_k}\}$, $|\varepsilon_{n_k}| \leqslant \varrho$ then we obtain

$$\operatorname{Re}\tilde{f}(z) \geqslant \frac{1-r}{1+r} - \varrho\sum_k |a_{n_k}| r^{n_k}$$

and Theorem 2 is proved. If we take the infimum of $\varphi(r)$ with respect to all functions $f(z)\in P$ then we have

$$\operatorname{Re}\tilde{f}(z) \geqslant \frac{1-r}{1+r} - 2\sum_k r^{n_k}|\varepsilon_{n_k}|.$$

This proves Theorem 3. If we take the infimum of $\varphi(r)$ with respect to all functions $f(z)\in P$ and all sequences $\{\varepsilon_{n_k}\}$ $|\varepsilon_{n_k}| \leqslant \varrho$ then we have

$$\operatorname{Re}\tilde{f}(z) \geqslant \frac{1-r}{1+r} - 2\varrho\sum_k r^{n_k}$$

and Theorem 4 is proved too. (We used here only the facts that if $f(z)\in P$ then $|a_n| \leqslant 2$ and $\operatorname{Re} f(z) \geqslant \dfrac{1-r}{1+r}$ for $|z| = r < 1$).

Let us put $f(z) = (1-z)/(1+z)$ and $\varepsilon_{n_k} = -\varrho(-1)^{n_k}$. Then for every given sequence $\{n_k\}$ we have

$$\tilde{f}(r) = f(r) + \sum_k \varepsilon_{n_k} a_{n_k} r^{n_k} = \frac{1-r}{1+r} - 2\varrho \sum_k r^{n_k}.$$

This proves that the results in Theorems 1, 2, and 3 are the best possible.

§ 4. Let S_0 be the class of functions of the form

(6) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

regular in the unit disc K_1 and such that

$$\operatorname{Re} f'(z) > 0.$$

S_0 is the class of functions with bounded rotation.

Now if we put $T = S_0$ then we obtain:

Theorem 5. *For fixed $\{n_k\}$, $\{\varepsilon_{n_k}\}$ and $f(z) \in S_0$ the number $R_1 = R_1(f, \{\varepsilon_{n_k}\}, S_0)$ is greater or equal to a positive root of the equation*

$$(1+r) \sum_k n_k |\varepsilon_{n_k} a_{n_k}| r^{n_k - 1} + r - 1 = 0.$$

Theorem 6. *For fixed $\{n_k\}$, ϱ and $f(z) \in S_0$ the number $R_2 = R_2(f, \{n_k\}, \varrho, S_0)$ is greater or equal to a positive root of the equation*

$$\varrho(1+r) \sum_k n_k |a_{n_k}| r^{n_k - 1} + r - 1 = 0.$$

There exist such functions $f(z) \in S_0$ and such sequences $\{\varepsilon_{n_k}\}$ that the results in Theorems 5 and 6 are the best possible.

Theorem 7. *For fixed $\{n_k\}$ and $\{\varepsilon_{n_k}\}$ the number $R_3 = R_3(\{\varepsilon_{n_k}\}, S_0)$ is greater or equal to a positive root of the equation*

$$2(1+r) \sum_k |\varepsilon_{n_k}| r^{n_k - 1} + r - 1 = 0.$$

Theorem 8. *For fixed $\{n_k\}$ and ϱ the number $R_4 = R_4(\{n_k\}, \varrho, S_0)$ is equal to a positive root of the equation*

$$2\varrho(1+r) \sum_k r^{n_k - 1} + r - 1 = 0.$$

The extremal function is $f(z) = -z + 2\log(1+z)$.

Proofs of Theorems 5–8. We can note that $f(z) \in S_0$ if and only if $f'(z) \in P$ and that if

$$\tilde{f}(z) = f(z) + \sum_k \varepsilon_{n_k} a_{n_k} z^{n_k}$$

then

$$\tilde{f}'(z) = f'(z) + \sum_k n_k \varepsilon_{n_k} a_{n_k} z^{n_k-1}.$$

It follows that

(7) $$R_1(f, \{\varepsilon_{n_k}\}, S_0) = R_1(f', \{\eta_{m_k}\}, P),$$

where

(8) $$\{m_k\} = \{n_k - 1\}, \ \{\eta_{m_k}\} = \{\varepsilon_{m_k+1}\} = \{\varepsilon_{n_k}\}.$$

Thus similary

(9) $$R_2(f, \{n_k\}, \varrho, S_0) = R_2(f', \{n_k - 1\}, \varrho, P),$$

(10) $$R_3(\{\varepsilon_{n_k}\}, S_0) = R_3(\{\eta_{m_k}\}, P),$$

and

(11) $$R_4(\{n_k\}, \varrho, S_0) = R_4(\{n_k - 1\}, \varrho, P).$$

Now we can obtain Theorems 5–8 from Theorems 1–4 by using (7)–(11).

Remark 8. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belongs to S_0 then the function $\tilde{f}(z) = z + \sum_{k=1}^{\infty} a_{2k} z^{2k}$ belongs to S_0 in K_r for every r such that $0 < r \leq \sqrt{2} - 1$. The result is the best. The extremal function is $f(z) = -z + 2\log(1+z)$.

Remark 9. $f(z) \in S_0$ then the function

$$f(z) = z + \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1} = \frac{1}{2}[f(z) - f(-z)]$$

belongs to S_0 too.

Remark 10. If $\{n_k\} = \{2k\}$ then

$$R_4(2k\}, 1, S_0) = 2 - \sqrt{3} = R_4(\{2k-1\}, 1, P).$$

If $\{n_k\} = \{2k+1\}$ then

$$R_4(\{2k+1\}, 1, S_0) = \sqrt{2} - 1 = R_4(\{2k\}, 1, P).$$

These remarks are implied by Remark 5.

5. It will be interesting to solve this problem for some other classes of regular functions: convex, starlike, close-to-convex, ...

We can also investigate a generalization of this problem, namely: Let T_1 and T_2 be two classes of regular functions in K_1. Let us suppose that $f(z) \in T_1$ and that $\tilde{f}(z)$ is given by (4).

To find the largest radius such that $\tilde{f}(z)$ belongs to T_2 in K_r.

For instance; let T_1 be the class of convex univalent functions and let T_2 be the class of starlike univalent functions. If we put $\{n_k\} = \{2k\}$, $\varepsilon_{n_k} = -1$ then we have

$$f(z) \in T_1 \Rightarrow \tilde{f}(z) \in T_2.$$

(See Nehari [2]).

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STRESZCZENIE

Niech T będzie ustaloną klasą funkcji regularnych w kole jednostkowym mających w nim rozwinięcie postaci

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots .$$

Niech dany będzie podciąg $\{n_k\}$ ciągu liczb naturalnych i ciąg $\{\varepsilon_{n_k}\}$ liczb zespolonych. Dla danej funkcji $f(z) \in T$ połóżmy

$$\tilde{f}(z) = \sum_{l=0}^{\infty} \tilde{a}_l z^l = f(z) + \sum_k \varepsilon_{n_k} a_{n_k} z^{n_k}$$

gdzie

$$\tilde{a}_l = \begin{cases} a_l & \text{jeżeli } l \neq n_k \\ n_{n_k} + \varepsilon_{n_k} a_{n_k} & \text{jeżeli } l = n_k \end{cases}$$

W pracy tej rozwiązywane są problemy wyznaczenia możliwie największej wartości $r (0 \leqslant r \leqslant 1)$ takiej, że

$$f(z) \in T \Rightarrow \tilde{f}(z) \text{ jest funkcją klasy } T \text{ w kole } |z| < r.$$

Problem ten został rozwiązany w przypadku gdy T jest klasą P funkcji Carathéodory'ego o części rzeczywistej dodatniej oraz gdy T jest klasą S_0 funkcji jednolistnych z ograniczonym obrotem.

РЕЗЮМЕ

Пусть T — Фиксированный класс регулярных функций в единичном круге, имеющий в этом круге следующее разложение:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots .$$

· Пусть $\{n_k\}$ -подпоследовательность последовательности натуральных чисел, $\{\varepsilon_{n_k}\}$ -последовательность комплексных чисел.

Для данной функции $f(z) \in T$ пусть

$$\bar{f}(z) = \sum_{l=0}^{\infty} \bar{a}_l z^l = f(z) + \sum_k \varepsilon_{n_k} a_{n_k} z^{n_k}$$

где

$$\bar{a}_l = \begin{cases} a_l, & l \neq n_k \\ a_{n_k} + \varepsilon_{n_k} a_{n_k}, & l = n_k . \end{cases}$$

В этой работе автор занимался определением максимального значения $r (0 \leqslant r \leqslant 1)$ так что $f(z) \in T \Rightarrow \bar{f}(z)$ является функцией класса Т в круге $|z| < r$.

Эта проблема была разрешима автором в случае, когда T является клас сом P функции Каратеодори с положительной вещественной частью а также когда T — класс S_0 однолистных функций с ограниченным поворотом.