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# Some Remarks Concerning Meromorphic Univalent Functions

Pewne uwagi dotyczące funkcji meromorficznych i jednolistnych

Некоторые заметки о мероморфных и однолистных функциях

## 1. Introduction

Let $U_p$, $0 < p < 1$, denote the family of functions $f(z) = z + a_2 z^2 + \ldots$, $|z| < p$, meromorphic and univalent in the unit disc $K_1 = \{z: |z| < 1\}$ which have a simple pole at the point $z = p$.

Let $R$ be the residue of a function $f \in U_p$ at the point $z = p$.

As pointed out by Y. Komatu [3] we have following estimates

$$p^2(1-p^2) \leqslant |R| \leqslant p(1-p^2)^{-1}$$

In view of this result the family $U_p$ is compact and can be investigated by variational methods.

We shall need the following

**Theorem A.** [4] *Suppose that $f \in U_p$, $z_k$ $(k = 1, 2, \ldots, m)$, $z_k \neq p$ are fixed points of $K_1$, $z_0$ satisfies $|z_0| = 1$, $A_k$ $(k = 1, 2, \ldots, m)$ are arbitrary complex numbers and $a = -p^{-1}R^{-1}$ Then there exists a positive number $\lambda_0$ such that for each $\lambda \in \langle 0, \lambda_0)$ there exist functions of the form*

$$(1) \quad f^*(z) = f(z) - \lambda \left\{ \sum_{k=1}^{m} A_k \left( \frac{z_k f'(z_k)}{f(z_k)} \right)^2 \frac{2f^2(z)}{f(z_k) - f(z)} + \right.$$

$$\left. + \sum_{k=1}^{m} A_k P(z, z_k) + \sum_{k=1}^{m} \bar{A}_k P(z, \bar{z}_k^{-1}) \right\} + O(\lambda^2),$$

$$(2) \quad f^{\Delta}(z) = f(z) + \lambda P(z, z_0) + O(\lambda^2)$$

*where*

$$P(z, u) = f(z) - zf'(z)\frac{u+z}{u-z} + af^2(z)\frac{p+u}{p-u}$$

*which belong to the class* $U_p$.

If the complementary set of the domain $f(K_1)$ has $w_k$ $(k = 1, 2, \ldots, m)$ as interior points, then there exist functions of the form

$$f^{**}(z) = f(z) - \lambda \sum_{k=1}^{m} A_k \frac{f^2(z)}{w_k - f(z)} + O(\lambda^2) \tag{3}$$

which belong to the family $U_p$.

Let $\Delta$ be the region of variability of the expression $\log \dfrac{zf'(z)}{f(z)}$, $z$ being fixed and $f$ ranging over the whole class $U_p$.

It is obvious that the set $\Delta$ is compact. Let $\partial\Delta$ be its boundary. A point $\mathscr{P}_0 \in \partial\Delta$ is said to be a regular boundary point if there exist a point $a \in \mathscr{C}\Delta$ and a disc $K(a, \varepsilon)$ such that

$$K(a, \varepsilon) \subset \mathscr{C}\Delta \quad \text{and} \quad \overline{K}(a, \varepsilon) \cap \partial\Delta = \{\mathscr{P}_0\}$$

It is well-known [5] that the set of regular boundary points is dense in $\partial\Delta$, moreover, the functions $f \in U_p$ corresponding to regular boundary points are extremal functions of the following extremal problem: $\min_{f \in U_p} |F(f) - a|$.

The main result obtained in this paper is Theorem 2 which determines the region $\Delta$. Our basic tool here is Theorem A.

It is my pleasant duty to express my thanks to Professor J. G. Krzyż for his helpful remarks.

## 2. A differential equation for extremal functions

Let $\Delta$ denote the set $\{w\colon\, w = \log \dfrac{zf'(z)}{f(z)}, f \in U_p\}$ and let $K(p)$, $\Pi(u, p)$ be elliptic integrals in Legendre's form, that is

$$K(p) = \int_0^1 [(1-x^2)(1-p^2x^2)]^{-1/2}\, dx,$$

$$\Pi(u, p) = \int_0^1 (1-ux^2)^{-1}[(1-x^2)(1-x^2p^2)]^{-1/2}\, dx$$

We are going to prove the following

**Theorem 1.** *Functions* $f(z, \varphi) = f_\varphi(z) \in U_p$, $\varphi \in \langle 0, 2\pi)$, *corresponding to regular boundary points of the set* $\Delta$ *map the disc* $K_1$ *to the whole plane* *slit along one analytic arc and satisfy the differential equation* ($|\zeta| < 1$):

$$(2.1)\quad e^{-i\varphi}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 \frac{f(z)f(\zeta)}{\big(f(\zeta)-f(z)\big)^2} = A\,\frac{\zeta(\zeta^2-(k+l)\zeta+kl)^2}{(\zeta-z)^2(1-\bar{z}\zeta)^2(\zeta-p)(1-p\zeta)}$$

*where* $k, l, |k| = |l| = 1$, *are determined by equations*

$$k+l = \bar{B}kl+B$$

$$(2.2)\qquad \frac{kl-(k+l)z+z^2}{\bar{z}^2kl-\bar{z}(k+l)+1} = e^{-i\varphi}\frac{z(p-z)(1-pz)}{|z(p-z)(1-pz)|}$$

$$(2.3)\qquad B = \frac{\Pi(p\bar{z}, p)+|z|^2\,\Pi(pz, p)-K(p)}{\bar{z}\big(\Pi(p\bar{z}, p)+\Pi(pz, p)-K(p)\big)}$$

$$(2.4)\qquad A = \frac{z(1-|z|^2)^2(p-z)(1-pz)}{(z-k)^2(z-l)^2}\,e^{-i\varphi}$$

$$(2.5)\qquad f(z) = \frac{pz(1-\bar{k}z)^2(1-\bar{l}z)^2}{(1-|z|^2)^2(p-z)(1-pz)}$$

**Proof.** Suppose that $\mathscr{I}_0$ is a regular boundary point of the set $\Delta$, $f \in U_p$ is an extremal function of the problem

$$\min_{f \in U_p} |F(f)-a|$$

where $a$ is a point mentioned above. Of cause, the function $f$ exists. Using the formula (3) we conclude after simple considerations, that the extremal functions map the unit disc onto domain $f(K_1)$ whose complementary set has no interior points.

Let us now apply the formula (1) with $m = 1$. Taking $\varphi = \operatorname{Arg}\times$ $\times(F(f)-a)$ we obtain after some calculus

$$|F(f^*)-a|^2 = |F(f)-a|^2+2\lambda|F(f)-a|\times$$
$$\times\mathscr{R}\left\{A_1\left[\left(e^{-i\varphi}\left(\frac{z_1f'(z_1)}{f(z_1)}\right)^2\frac{f(z)f(z_1)}{\big(f(z_1)-f(z)\big)^2}-\frac{\partial}{\partial z}P(z, z_1)-\right.\right.\right.$$
$$\left.\left.\left.-\overline{\frac{\partial}{\partial z}P(z, \bar{z}_1^{-1})}\right]\right\}+O(\lambda^2)$$

Because the argument of $A_1$ can be chosen in an arbitrary manner the term in braces must be equal to zero for all $z_1 \in K_1$. Taking $z_1 = \zeta$ we conclude that the extremal functions satisfy the following equation

$$(2.6)\quad e^{-i\varphi}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2\frac{f(z)f(\zeta)}{\big(f(\zeta)-f(z)\big)^2} = C\frac{\zeta+z}{\zeta-z}+\bar{C}\frac{1+\bar{z}\zeta}{1-\bar{z}\zeta}+D\frac{\zeta+z}{\zeta-z}+$$
$$+\frac{1+\bar{z}\zeta}{1-\bar{z}\zeta}\bar{D}+\frac{\bar{z}\zeta}{(1-\bar{z}\zeta)^2}e^{i\varphi}+\frac{z\zeta}{(\zeta-z)^2}e^{-i\varphi} = Q(\zeta)$$

where

$$2C = e^{-i\varphi}\left(\frac{\big(zf'(z)\big)'}{f'(z)} - \frac{zf'(z)}{f(z)}\right);\ 2D = -af(z)e^{-i\varphi}$$

It is easy to see, that for all $\zeta \in K_1$ we have

$$\overline{Q(\zeta)} \equiv Q(\bar{\zeta}^{-1}) \tag{2.7}$$

We shall prove now that

$$Q(\zeta) \leqslant 0 \quad \text{for} \quad |\zeta| = 1. \tag{2.8}$$

In order to prove it we use the formula (2) and we obtain the following condition

$$|F(f^{\Delta})-a|^2 = |F(f)-a|^2 -$$

$$\lambda\mathscr{R}\left\{C\frac{z_0+z}{z_0-z} + D\frac{z_0+p}{z_0-p} + \frac{z_0 z}{(z_0-z)^2}e^{-i\varphi}\right\}|F(f)-a| + O(\lambda^2)$$

However, $f$ is an extremal function, thus

$$\mathscr{R}\left(C\frac{z_0+z}{z_0-z} + D\frac{z_0+p}{z_0-p} + \frac{z_0 z}{z_0-z)^2}e^{-i\varphi}\right) \leqslant 0$$

and our statement is proved.

The function $Q(\zeta)$ has zeros for $|\zeta| < 1$ only at the point $\zeta = 0$. The condition (2.7) implies that $Q(\zeta)$ has no zeros for $\zeta \neq 0$ and $|\zeta| \neq 1$. Moreover, the condition (2.8) shows that the zeros of $Q(\zeta)$ on the unit circle have multiplicity of even order. Now, $Q(\zeta)$ is a rational function whose numerator is a polynomial of degree $\leqslant 6$. The considerations made above imply that its degree is actually 5.

Let $k, l$ be zeros of $Q(\zeta)$ on $|\zeta| = 1$ (which correspond to the roots of the equation $f'(\zeta) = 0$). The equation (2.6) takes now the form

$$e^{-i\varphi}\left(\frac{(\zeta f'(\zeta)}{f(\zeta)}\right)^2 \frac{f(z)f(\zeta)}{\big(f(\zeta)-f(z)\big)^2} = A\,\frac{\zeta(\zeta-k)^2(\zeta-l)^2}{(\zeta-z)^2(1-\bar{z}\zeta)^2(\zeta-p)(1-p\zeta)} \tag{2.9}$$

where $A$ is a constant.

### Determining of constants $A, f(z), k, l$

It follows from previous considerations that the extremal functions map the unit disc onto the whole plane slit along one analytic arc. There are two arcs $l_1, l_2$ on $|\zeta| = 1$ with common end points $k, l$ which are carried by $f$ into both edges of the slit. If $\zeta_1 \in l_1$, $\zeta_2 \in l_2$ are points corresponding to the same point $w$ on the slit $f(C_1)$ then (cf. [5], p. 112–117)

$$\int_{l_1} Q(\zeta_1)d\zeta_1 = \int_{l_2} Q(\zeta_2)d\zeta_2$$

Hence, taking $\zeta = e^{i\varepsilon}$, $k = e^{i\alpha}$, $l = e^{i\beta}$, we have

$$\int_{\alpha}^{\beta} |Q(e^{i\theta})|^{1/2} d\theta = \int_{\beta}^{\alpha+2\pi} |Q(e^{i\theta})|^{1/2} d\theta$$

However,

$$|Q(e^{i\theta})|^{1/2} = 4\left|\sin\frac{\theta-\alpha}{2}\right|\left|\sin\frac{\theta-\beta}{2}\right| |z-e^{i\theta}|^{-2}|p-e^{i\theta}|^{-1}$$

and due to the periodicity of the function $Q(e^{i\theta})$, we obtain ultimately

$$\int_0^{2\pi} q(e^{i\theta})\sin\frac{\theta-\alpha}{2}\sin\frac{\theta-\beta}{2}\,d\theta = 0$$

where $q(e^{i\theta}) = |z-e^{i\theta}|^{-2}|p-e^{i\theta}|^{-1}$ and also

(2.10) $$k+l = \bar{B}kl+B$$

where

(2.11) $$B = \int_0^{2\pi} e^{-i\theta}q(e^{i\theta})d\theta \Big/ \int_0^{2\pi} q(e^{i\theta})d\theta.$$

As shown above, we have $Q(e^{i\theta}) \leqslant 0$ for real $\theta$, hence $\sqrt{Q(\zeta)}$ is purely imaginary on the unit circle. Put $A = |A|e^{i\varphi}$. The condition $\mathscr{R}\sqrt{Q(\zeta)} = 0$ for $|\zeta| = 1$ yields

$$\mathscr{R}\{e^{i\varphi/2}[\zeta+kl\zeta^{-1}-(k+l)]\} = 0$$

and thus

(2.12) $$kl = -e^{-i\varphi}$$

Comparing Laurent's coefficient of both sides in (2.8) we obtain

(2.13) $$A = \frac{z(1-|z|^2)^2(p-z)(1-pz)}{(z-k)^2(z-l)^2}e^{-i\varphi}$$

(2.14) $$f(z) = zp(1-zk)^2(1-zl)^2(1-z^2)^2(p-z)^{-1}(1-pz)^{-1}$$

Now (2.12) and (2.13) yield the condition

$$\mathscr{R}\{e^{-i\varphi}(z-p)(1-pz)(z-k)^{-2}(z-l)^{-2}kl\} = 0$$

and from this we have

(2.15) $$\frac{z-k}{1-k\bar{z}}\,\frac{z-l}{1-l\bar{z}} = e^{i\varphi}\,\frac{z(p-z)(1-pz)}{|z(p-z)(1-pz)|}$$

In order to complete the proof of Th. 1 we shall bring the elliptic integrals in the formula (2.11) to the normal form.

Putting $t = e^{i\theta}$ we have ($C_1 = \{\zeta: |\zeta| = 1\}$)

$$B = \int_{C_1} v(t)(t-z)^{-1}(1-t\bar{z})^{-1}dt \Big/ \int_{C_1} tv(t)(t-z)(1-t\bar{z})^{-1}dt$$

$$= \Big[\int_{C_1}(1-tz)^{-1}v(t)\,dt + |z|^2\int_{C_1}(1-tz)^{-1}v(t)\,dt -$$

$$- \int_{C_1} v(t)\,dt\Big]\, z^{-1}\Big[\int_{C_1}(1-tz)^{-1}v(t)\,dt + \int_{C_1}(1-t\bar{z})^{-1}v(t)\,dt - \int_{C_1} v(t)\,dt\Big]^{-1}$$

where $v(t) = [t(p-t)(1-pt)]^{-1/2}$.

Let us slit the $t$ — plane along the segments $[0, p]$ and $[p^{-1}, \infty]$. All functions under the signs of integrals are regular in the slit domain and the paths of integration can be continuously deformed to the segments $[0, p]$. Hence, we obtain

$$B = \Big[\int_0^p (1-tz)^{-1}v(t)\,dt + |z|^2\int_0^p (1-t\bar{z})^{-1}v(t)\,dt -$$

$$- \int_0^p v(t)\,dt\Big]\cdot z^{-1}\Big[\int_0^p (1-t\bar{z})^{-1}v(t)\,dt - \int_0^p v(t)\,dt + \int_0^p (1-tz)^{-1}v(t)\,dt\Big]^{-1}.$$

Using the substitution $t = p\zeta^2$ we bring $B$ to the form given in (2.4) and Th. 1 is completely proved.

## 3. The region $\Delta$

We shall use the previous results to prove following

**Theorem 2.** *If $z, z \neq p$, is a fixed point of the disc $K_1$ and $f$ range over the whole class $U_p$ then the region $\Delta$ is a disc given by the equation*

$$\left|F(f) - \frac{F_1 - F_2}{2}\right| \leqslant \left|\frac{F_1 + F_2}{2}\right| \tag{3.1}$$

*where $F_1 = F(f(z, 0))$, $F_2 = F(f(z, \pi))$.*

**Proof.** It follows from the equation (2.1) that for each function which satisfies (2.1) and for an arbitrary branch of root we have the condition

$$\mathscr{R}\left\{e^{-i\varphi/2}\int \left(w^{-1}f(z)\right)^{1/2}\left(w-f(z)\right)^{-1}dw\right\} = \text{const}$$

on the $|\zeta| = 1$, ($w = f(\zeta)$), which is equivalent to

$$\mathscr{R}\left\{e^{-i\varphi/2}\log\frac{\sqrt{f(\zeta)}-\sqrt{f(z)}}{\sqrt{f(\zeta)}+\sqrt{f(z)}}\right\} = C. \tag{3.2}$$

The functions $f_1 = f(z,0), f_2 = f(z,\pi)$ satisfy (3.2) for $\varphi = 0, \varphi = \pi$ respectively, hence

$$\mathscr{R}\left\{\log\frac{\sqrt{f_1(\zeta)}-\sqrt{f_1(z)}}{\sqrt{f_1(\zeta)}+\sqrt{f_1(z)}}\right\} = C_1 \tag{3.3}$$

$$\mathscr{R}\left\{-i\log\frac{\sqrt{f_2(\zeta)}-\sqrt{f_2(z)}}{\sqrt{f_2(\zeta)}+\sqrt{f_2(z)}}\right\} = C_2 \tag{3.4}$$

We have now

$$\mathscr{R}\left\{\cos\varphi/2\log\frac{\left(\sqrt{f(\zeta)}-\sqrt{f(z)}\right)\left(\sqrt{f_1(\zeta)}+\sqrt{f_1(z)}\right)}{\left(\sqrt{f_1(\zeta)}-\sqrt{f_1(z)}\right)\left(\sqrt{f(\zeta)}+\sqrt{f(z)}\right)}-\right. \tag{3.5}$$

$$\left.-i\sin\varphi/2\log\frac{\left(\sqrt{f(\zeta)}-\sqrt{f(z)}\right)\left(\sqrt{f_2(\zeta)}+\sqrt{f_2(z)}\right)}{\left(\sqrt{f_2(\zeta)}-\sqrt{f_2(z)}\right)\left(\sqrt{f(\zeta)}+\sqrt{f(z)}\right)}\right\} = \text{const}.$$

The functions $f, f_1, f_2$ are holomorphic in the domain $K_1 \setminus \{p\}$ and map the unit disc onto an analytic arc, hence they are continuous on the unit circle. The function $G(\zeta)$ under the sign of real part in (3.5) is holomorphic at the point $\zeta = z$ and has two branch points ($\zeta = 0, \zeta = p$) in which it has finite values, moreover, it has a constant real part on the unit circle and it is continuous there. If we consider this function on a double-sheeted Riemann surface consisting of two unit discs with branch points $0, p$, then the function will be single-valued. This Riemann surface is conformally equivalent to an annulus. However, any single-valued function continuous in a closed annulus which has a constant real part on its boundary, is a constant. Hence $G(\zeta) \equiv \text{const}$. In view of $\lim_{\zeta=0} G(\zeta) = 0$, we conclude, that $G(\zeta) \equiv 0$.

On the other hand, taking $\zeta = z$ we have

$$e^{-i\varphi/2}\log\frac{zf'(z)}{f(z)} - \cos\frac{\varphi}{2}\log\frac{zf_1'(z)}{f_1(z)} + i\sin\frac{\varphi}{2}\log\frac{zf_2'(z)}{f_2(z)} = 0$$

hence

$$\log\frac{zf'(z)}{f(z)} - 2^{-1}\left(\log\frac{zf_1'(z)}{f_1(z)} - \log\frac{zf_2'(z)}{f_2(z)}\right) = \tag{3.6}$$

$$= 2^{-1}\left(\log\frac{zf_1'(z)}{f_1(z)} + \log\frac{zf_2'(z)}{f_2(z)}\right)e^{i\varphi}$$

and Th. 2 is proved.

Obviously, the result (3.6) can be also obtained by integration of the equation (2.1). In this case we obtain the centre and the radius of the disc (3.6) in terms of functions of some elliptic integrals however, the corresponding formulas are involved. The main idea of the proof resembles a method used by Golusin (cf. [1] p. 127–139) and seems to be more simple than the immediate integration of the equation (2.1).

## 4. Some particular cases

$1^0$ Let us first consider the case $p \to 1$. It follows immediately from (2.5) that

$$\lim_{p=1}(B(z,p)-1) = 0 \tag{4.1}$$

and the equations (2.2) give us

$$k = 1,\ l = z|z|^{-1}(|z|+e^{i\varphi})(1+|z|e^{i\varphi})^{-1}$$

and (2.1) yields

$$\left|\log\frac{zf'(z)}{f(z)}\right| \leqslant \log(1+|z|)(1-|z|)^{-1}$$

which is the well-known region of variability of $\log\dfrac{zf'(z)}{f(z)}$ in the family $S$ (cf. [2]).

$2^0$ Let us take $z = \bar{z}$. Then $B = \bar{B}$ and equations (2.2) yield

$$k+l = B(1+kl)$$

$$\frac{k-z}{1-kz}\,\frac{l-z}{1-lz} = e^{i\varphi}\frac{z(p-z)(1-pz)}{|z(p-z)(1-pz)|}.$$

If $0 < z < p$ and $\varphi = \pi$, or $-1 < z < 0, \varphi = 0$ then

$$k+l = B(1+kl)$$

$$2z(k+l) = (1+|z|^2)(1+kl)$$

and because $B \leqslant 1$ we have $k = -l = +1$ and (2.1) can be integrated in an elementary way. Its solution is

$$f(z) = pz(p-z)^{-1}(1-pz)^{-1}\in U_p.$$

Hence we have the sharp estimate

$$\left|\frac{zf'(z)}{f(z)}\right| \leqslant p(1-z^2)(p-z)^{-1}(1-pz)^{-1}$$

for $z\in(0,p)$.

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## STRESZCZENIE

W pracy tej określono zbiór wszystkich możliwych wartości funkcjonału $\log(zf'(z)/f(z))$ w klasie $U(p)$.

## РЕЗЮМЕ

В работе определено множество всех возможных значений функционала $\log(zf'(z)/f(z))$ в классе $U(p)$.