ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XX, 3 SECTIO A 1966

Z Katedry Funkcji Analitycznych Wydz. Mat. Fiz. Chem. UMCS
Kierownik: prof. dr Jan Krzyż

JACEK CHĄDZYŃSKI AND JULIAN ŁAWRYNOWICZ

On Homeomorphisms and Quasiconformal Mappings Connected with Cyclic Groups of Homographies and Antigraphies

O homeomorfizmach i odwzorowaniach quasi-konforemnych związanych z grupami cykliczynymi homografii i antygrafii

О гомеоморфных и квазиконформных отображениях, связанных с циклическими группами гомографических и антиграфических преобразований

1. Notation

Throughout this paper we are always concerned with points and sets on the closed plane $\mathscr{E}$. The difference of two sets E and E' is denoted by $E\setminus E'$, the closure of E by $\operatorname{cl} E$, the interior of E by $\operatorname{int} E$, and the boundary of E by $\operatorname{fr} E$. We assume that $z/0 = \infty$ for $z \in \mathscr{E}\setminus\{0\}$, and $z/\infty = 0$ for $z \in \mathscr{E}\setminus\{\infty\}$. Next let

$$\Delta_t = \{z\colon t \leqslant |z| \leqslant 1\},\ \Delta_t^* = \{z\colon t \leqslant |z| \leqslant 1/t\} \text{ for } 0 \leqslant t < 1.$$

Under Jordan curve we mean a homeomorphic image of a circle, under Jordan arc — a homeomorphic image of an interval, i.e. of a connected subset of the open straight line, which does not reduce to a point.

If f is a function defined on E, and $E' \subset E$, then $f[E']$ denotes the image of E' under f. If, in particular, f is an elementary function: exp, arg etc., and $z \in E$, we write fz instead of $f(z)$ in case where it does not lead to misunderstanding. We say that f, defined on E, satisfies a property, if this property is satisfied for all $z \in E$. If f and g are functions defined on E and E', respectively, where $E' \supset f[E]$, then the composite function defined on E is denoted by $g\circ f$, and for $z \in E$ we write $g\circ f(z)$ instead of $(g\circ f)(z)$. Further, $f\colon E \to E'$ means that f is a mapping of E *onto* E', and $\check{f}$ denotes the inverse of f, if it exists, while $f^{-1} = 1/f$. The notation $\check{f}$ for the inverse of f is used e.g. in [4] and is much more convenient for our purpose than f^{-1}. Finally, if f is a function of a complex variable $z = x+iy$, we denote its partial derivatives, if they exist, by

f'_x and f'_y, while $f'_z = \frac{1}{2}(f'_x - if'_y)$ and $f'_{\bar{z}} = \frac{1}{2}(f'_x + if'_y)$ are the *formal complex derivatives* of f. In particular, we denote f'_z by f' when f has the total differential and $f'_{\bar{z}} = 0$ (cf. e.g. [14], p. 59). Symbols f_τ, f_k etc. denote functions depending on some parameters τ, k etc.

The expression *if and only if* is abbreviated by *iff*, while the expression *almost everywhere* by *a.e.*

2. Introduction

Let h be an arbitrary fixed homography (synonyms: homographic transformation, bilinear transformation, (fractional) linear transformation, Möbius transformation) which is not loxodromic, and let a be an arbitrary fixed antigraphy (a synonym: anti-homography). For definitions and properties of these transformations we refer to [4] and [15]. Throughout the paper we assume that h is not the identity mapping, and a is not a Möbius involution. It is well known that $a \circ a$ is a homography which is not loxodromic, and that given h there is an antigraphy $\tilde{a}$ such that $\tilde{a} \circ \tilde{a} = h$.

Let n, $n \neq 1$, be a positive integer. Consider a homography that is not loxodromic and generates an n-cyclic group (of homographies) with respect to composition. It is well known (cf. e.g. [14], pp. 86-87) that h must be elliptic and that, given s_0 different from the invariant points of h, the points $s_k = h(s_{k-1})$, $s_{-k} = \check{h}(s_{-k+1})$, $k = 1, 2, \ldots$, and s_0 satisfy $s_{k+n} = s_k$ for $k = 0, 1, -1, 2, -2, \ldots$, and lie either on a circle or on a straight line. In case where h is elliptic and does not generate a cyclic group with respect to composition, the points s_k, $k = 0, 1, -1, 2, -2, \ldots$, form a dense subset of either a circle or a straight line. Thus, being interested in the cyclic cases, we shall consider problems formulated below in case of an arbitrary elliptic homography or antigraphy and show that we have either an *n-cyclic case* or a *limit case*.

After these preliminary remarks we are able to formulate the problems to be discussed in this paper. Given an elliptic homography h and an elliptic antigraphy a we are concerned with studying homeomorphic solutions of the functional equation

$$g \circ h(s) = h \circ g(s) \tag{1}$$

and homeomorphic solutions of

$$g \circ a(s) = a \circ g(s), \tag{2}$$

which map some D onto D', where D is a given domain or the closure of a domain, both bounded by disjoint Jordan curves. In particular, we are concerned with Q-quasiconformal solutions of (1) and (2), subject to suitable conditions in order to assure that two Q-quasiconformal mappings which satisfy the same equation, are identical. Here $1 \leqslant Q < +\infty$.

For a definition and properties of quasiconformal mappings we refer to [8]. We notice that under a Q-quasiconformal mapping of the closure of a domain bounded by disjoint Jordan curves we understand any homeomorphism which is a Q-quasiconformal mapping of $\operatorname{int} D$.

It is clear that in case of (1) we have to assume that $D = h[D]$. Hence $\operatorname{fr} D$ is the union of some sets being connected subsets of straight lines or circles, which do not reduce to a point. Consequently (cf. [8], p. 44), any Q-quasiconformal solution of (1), determined in $\operatorname{int} D$, can be continued to a Q-quasiconformal mapping of $\operatorname{cl} D$. Thus we confine ourselves to closed domains. A closed domain bounded by disjoint Jordan curves is said to be a *natural domain* with respect to a homography h if $D = h[D]$.

Similarly, a closed domain $\tilde{D}$ bounded by disjoint Jordan curves is said to be a *natural domain* with respect to an antigraphy a if $\tilde{D} = a \circ a[\tilde{D}]$, $\tilde{D} \cap a[\tilde{D}] \subset \operatorname{fr} \tilde{D}$, and $\tilde{D} \cup a[\tilde{D}]$ is a closed domain. Clearly, we may confine ourselves to consider homeomorphic solutions of (2) determined on sets of the form $D = \tilde{D} \cup a[\tilde{D}]$, where $\tilde{D}$ is natural. It can easily be verified that if a homeomorphic solution of (2), defined on D, satisfies $g[\tilde{D}] = \tilde{D}'$, then $g[D] = \tilde{D}' \cup a[\tilde{D}']$.

The problems discussed in this paper were posed in [12], where there was also introduced the notion of natural domain (in [12], p. 344, line 18, it should be assumed that D is invariant under $w = \overline{h \circ h(\bar{z})}$).

We begin our considerations with obtaining relations between homeomorphic solutions of (1) and (2) (Lemma 1), and then reduce the problems in question to analogous problems with some normalized h and a (Lemma 2). Then, according to suggestions given in [12], we distinguish the cyclic case and the limit case, and consider them separately. In each case we extend Lemma 2 (Lemmas 3 and 4, respectively), and then strengthen Lemma 1 (Theorems 1 and 4, respectively, which include also results formulated in the lemmas). Next we confine ourselves to quasiconformal mappings, normalize them as suggested in [12] (pp. 344-345), and obtain some relations between the normalized classes (Theorems 2 and 5). Finally we characterize the classes under consideration in terms of complex dilatation (Theorems 3 and 6). This is a generalization of some results obtained in [12].

The results of this paper were announced in [5] and [6].

3. Relations between homeomorphic solutions of (1) and (2) in the general case

In this section we obtain a generalization of Lemma 3 and Remark 7, both given in [12] (p. 336). Consider an elliptic homography:

(3) $$h(s) = \check{r} \circ h^* \circ r(s),$$

where

(4) $$h^*(z) = e^{ia}z, \; -\infty < a < +\infty, \; e^{ia} \neq 1,$$

(5) $$r(s) = \begin{cases} (s-s_1)/(s-s_2) & \text{for } s_1, s_2 \neq \infty, \\ s-s_1 & \text{for } s_1 \neq \infty, s_2 = \infty. \end{cases}$$

Assume that the natural domain is of the form

$$D = \{s\colon \tau_0 \leqslant |r(s)| \leqslant \tau\}, \text{ where } 0 \leqslant \tau_0 < \tau < +\infty.$$

Suppose first that $\tau \neq +\infty$ and denote by D' any domain of the form

$$D' = \{v\colon \tau_0' \leqslant |r(v)| \leqslant \tau\}, \text{ where } 0 \leqslant \tau_0' < \tau,$$

and $\tau_0' = 0$ iff $\tau_0 = 0$. Consider the antigraphy

(6) $$a_\tau(s) = \check{r}_\tau \circ a^* \circ r_\tau(s),$$

where

(7) $$a^*(z) = e^{(1/2)ia}/\bar{z},$$

(8) $$r_\tau(s) = \tau^{-1} r(s),$$

and equation (2) with a_τ substituted for a:

(9) $$g \circ a_\tau(s) = a_\tau \circ g(s).$$

Lemma 1. *(i) If a homeomorphism $g\colon D \to D'$ is a solution of* (1), *and* (9) *holds for $g|D \cap a_\tau[D]$, then g^*, defined by*

(10) $$g^*|D = g,$$

(11) $$g^*|a_\tau[D] \setminus D = a_\tau \circ g \circ \check{a}_\tau|a_\tau[D] \setminus D,$$

is a homeomorphic solution of (1), *and*

(12) $$g^*\big[D \cup a_\tau[D]\big] = D' \cup a_\tau[D'].$$

(ii) If a homeomorphism $g\colon D \cup a_\tau[D] \to D' \cup a_\tau[D']$ is a solution of (9), *then*

(13) $$g_1 = g|D, \; g_2 = g|a_\tau[D]$$

are homeomorphic solutions of (1). *If, in addition, $|r \circ g(s)| = \tau^*$ for $s \in D \cap a_\tau[D]$, then $\tau^* = \tau$, i.e. either*

(14) $$g_1[D] = D', \; g_2 \circ a_\tau[D] = a_\tau[D']$$

or

(15) $$g_1[D] = a_\tau[D'], \; g_2 \circ a_\tau[D] = D'.$$

Proof. We start with proving (i). Let $g\colon D \to D'$ be a homeomorphic solution of (1). If $s \in D \setminus a_\tau[D]$ then, clearly, $a_\tau(s) \in a_\tau[D] \setminus D$. Thus, by (11), for $s \in D \setminus a_\tau[D]$ we have $g^* \circ a_\tau(s) = a_\tau \circ g(s)$. Applying now (10) we get $g^* \circ a_\tau(s) = a_\tau \circ g^*(s)$ for $s \in D \setminus a_\tau[D]$. Next, if $s \in a_\tau[D] \setminus D$ then, clearly, $a_\tau(s) \in D \setminus a_\tau[D]$. Hence, by (10), for $s \in a_\tau[D] \setminus D$ we have $g^* \circ a_\tau(s) = g \circ a_\tau(s)$. But $a_\tau = a_\tau \circ a_\tau \circ \check{a}_\tau$. Applying the relations $a_\tau \circ a_\tau = h$ and (1) we get $g^* \circ a_\tau(s) = a_\tau \circ a_\tau \circ g \circ \check{a}_\tau(s)$. Therefore, by (11), for $s \in a_\tau[D] \setminus D$ we obtain $g^* \circ a_\tau(s) = a_\tau \circ g^*(s)$ as well. Finally, if we assume that (9) holds for $s \in D \cap a_\tau[D]$ then, by (10), g^* is a solution of (9).

We claim that g^* is a homeomorphism. Indeed, by (10), $g^*|D$ is a homeomorphism. Therefore, since g^* is a solution of (9), and a_τ is a homeomorphism, $g^*|a_\tau[D]$ is a homeomorphism as well and, by (10),

$$g^*|a_\tau[D] = a_\tau \circ g \circ \check{a}_\tau | a_\tau[D]. \tag{16}$$

Hence we obtain two conclusions. At first, g^* is continuous. Since it is defined on $D \cup a_\tau[D]$ which is closed, we have only to prove that g^* is one-one. At second, since, by (10),

$$g^*[D] = D', \tag{17}$$

and, by (16),

$$g^* \circ a_\tau[D] = a_\tau[D'], \tag{18}$$

then in order to prove that g^* is one-one it is enough to show that

$$g^*[D \cap a_\tau[D]] = D' \cap a_\tau[D'].$$

But this follows from the relation

$$g^*[D \cap a_\tau[D]] \subset g^*[D] \cap g^* \circ a_\tau[D] = D' \cap a_\tau[D'],$$

which is itself a consequence of (17) and (18), and from the fact that any homeomorphism maps connected sets onto connected sets. Thus g^* is a homeomorphism, as desired. Besides, (17) and (18) imply (12).

We proceed to prove (ii). Let $g\colon D \cup a_\tau[D] \to D' \cup a_\tau[D']$ be a homeomorphic solution of (9). Next let $s \in D$. Clearly, also $h(s) \in D$. Hence, by (13) and $h = a_\tau \circ a_\tau$, we have $g_1 \circ h(s) = g \circ a_\tau \circ a_\tau(s)$. Applying now relation (9) twice, we obtain $g_1 \circ h(s) = a_\tau \circ a_\tau \circ g(s)$. Using again $a_\tau \circ a_\tau = h$ and (13), we get $g_1 \circ h(s) = h \circ g_1(s)$. Thus g_1 is a homeomorphic solution of (1), as desired. Since g is a homeomorphism, so is $g_1 = g|D$. Analogously we prove that g_2 is a homeomorphic solution of (1) as well.

Suppose now, in addition, that $|r \circ g(s)| = \tau^*$ for $s \in D \cap a_\tau[D]$. Take an arbitrary $s \in D \cap a_\tau[D]$, i.e. an arbitrary s satisfying $|r(s)| = \tau$. By (6), (8) and (7) we have

$$|r \circ a_\tau(s)| = |r \circ \check{r}_\tau \circ a^* \circ r_\tau(s)| = |\tau a^*(\tau^{-1} r(s))| = |\tau^2/\overline{r(s)}| = \tau.$$

Hence, according to the additional assumption,

$$|r\circ g\circ a_\tau(s)| = \tau^*.$$

On the other hand, by (6), (8), (7) and the additional assumption, we have

$$|r\circ a_\tau\circ g(s)| = |r\circ \check{r}_\tau\circ a^*\circ r_\tau\circ g(s)| = |\tau a^*(\tau^{-1} r\circ g(s))| = |\tau^2/\overline{r\circ g(s)}| = \tau^2/\tau^*.$$

Consequently, by (9), we obtain $\tau^* = \tau^2/\tau^*$, i.e. $\tau^* = \tau$. Since g is a homeomorphism, and g_1, g_2 are defined by (13), this means that we have either (14) or (15). The proof of Lemma 1 is completed.

If D' is of the form $D' = \{v\colon \tau \leqslant |r(v)| \leqslant \tau_0'\}$, where $\tau < \tau_0' \leqslant +\infty$, and $\tau_0' = +\infty$ iff $\tau_0 = 0$, then statements analogous to that of Lemma 1 hold.

In Lemma 1 we have assumed that $\tau \neq +\infty$. Now, let us replace this condition with $\tau_0 \neq 0$. Then, clearly, statements analogous to that of Lemma 1 hold.

Finally suppose that $\tau_0 = 0$ and $\tau = +\infty$, i.e. $D = \mathscr{E}$. Since we consider homeomorphic solutions of (1), $D' = \mathscr{E}$ as well. Let $\tilde{\Delta}_t = \{z\colon |z| \leqslant t\}$ for $0 < t < +\infty$. We have:

Remark. (i) If a homeomorphism $g\colon \mathscr{E}\to\mathscr{E}$ is a solution of (1), $g\circ\check{r}[\tilde{\Delta}_t] = \check{r}[\tilde{\Delta}_t]$ for some t, $0 < t < +\infty$, and (9) with $\tau = t$ holds for $g|\check{r}[\operatorname{fr}\tilde{\Delta}_t]$, then we can apply Lemma 1 (i) to $g|\check{r}[\tilde{\Delta}_t]$. An analogous statement holds for $g|\check{r}[\mathscr{E}\setminus\operatorname{int}\tilde{\Delta}_t]$, and in case where the condition $g\circ\check{r}[\tilde{\Delta}_t] = \check{r}[\tilde{\Delta}_t]$ is replaced with $g\circ\check{r}[\tilde{\Delta}_t] = \check{r}[\mathscr{E}\setminus\operatorname{int}\tilde{\Delta}_t]$.

(ii) If a homeomorphism $g\colon \mathscr{E}\to\mathscr{E}$ is a solution of (9) with $\tau = t$, where $0 < \tau < +\infty$, then we can apply Lemma 1 (ii) with $D = \check{r}[\tilde{\Delta}_t]$ to g. An analogous statement holds for $D = \check{r}[\mathscr{E}\setminus\operatorname{int}\tilde{\Delta}_t]$.

4. The problems with normalized h and a

Lemma 1 shows that the problems of finding homeomorphic solutions of (1) and (2) are, in general, not equivalent. In Section 6 it will be shown that they are not equivalent even in case of quasiconformal solutions. In order to give further details concerning the problem in question, we transform (1) and (2) to a normalized form. This is given by the following obvious lemma:

Lemma 2. (*i*) *The problem of solving* (1) *in a natural domain D is equivalent to the problem of solving*

$$f\circ h^*(z) = h^*\circ f(z) \tag{19}$$

in $r_\tau[D]$, where $f = r_\tau\circ g\circ\check{r}_\tau$ and $z = r_\tau(s)$ for $s\in D$.

(ii) *The problem of solving* (9) *in* $D \cup a_\tau[D]$, *where* D *is a natural domain, is equivalent to the problem of solving*

$$f \circ a^*(z) = a^* \circ f(z) \tag{20}$$

in $r_\tau[D \cup a_\tau[D]]$, *where* $f = r_\tau \circ g \circ \check{r}_\tau$ *and* $z = r_\tau(s)$ *for* $s \in D \cup a_\tau[D]$.

In Lemma 2 we have considered (9) instead of (2) since every elliptic antigraphy can be written in the form (6).

Owing to Lemma 2 and reasons given in Section 2 (also in [12], pp. 344-345), we shall consider, separately, the cases:

(I) $h = h^*$ and $a = a^*$, a/π rational, $e^{ia} \neq 1$,

(II) $h = h^*$ and $a = a^*$, a/π irrational,

called the *cyclic case* and the *limit case*, respectively. In both cases we shall distinguish two particular cases:

(a) $D = \Delta_t$, where $0 \leqslant t < 1$,

(b) $D = \mathscr{E}$.

The particular cases corresponding to (b) are called *continued* for the reasons explained by the Remark (also by Lemma 3 and Remark in [12], p. 336).

I. The cyclic case

5. Homeomorphic solutions

In the case under consideration the problems in question can be simplified again.

Let n, $n \neq 1$, be a positive integer. Further let k be an integer such that k and n are relatively prime.

Lemma 3. (*i*) *A homeomorphism* $f\colon \Delta_t \to \Delta_t$ (*or* $f\colon \mathscr{E} \to \mathscr{E}$) *is a solution of* (19) *with* $a = 2k\pi/n$ *iff it is a solution of* (19) *with* $a = 2\pi/n$.

(*ii*) *A homoeomorphism* $f\colon \Delta_t^* \to \Delta_{t'}^*$ *is a solution of* (20) *with* $a = 4k\pi/n$ *iff it is a solution of* (20) *with* $a = 4\pi/n$.

Proof. Clearly, if a homeomorphism $f\colon \Delta_t \to \Delta_{t'}$ (or $f\colon \mathscr{E} \to \mathscr{E}$) is a solution of (19) with $a = 2\pi/n$, then it is a solution of (19) with $a = 2k\pi/n$.

Conversely, suppose that a homeomorphism $f\colon \Delta_t \to \Delta_{t'}$ (or $f\colon \mathscr{E} \to \mathscr{E}$) is a solution of (19) with $a = 2k\pi/n$. Since k and n are relatively prime, there exists a pair of integers k_0 and n_0 such that $k_0 k + n_0 n = 1$, i.e. $k_0 k/n = -n_0 + 1/n$. Hence f is a solution of (19) with $a = 2\pi/n$.

Suppose now that a homeomorphism $f\colon \Delta_t^* \to \Delta_{t'}^*$ is a solution of (20) with $a = 4\pi/n$. If k is odd the assertion is obvious, so we may assume that k is even. This implies that $f|\Delta_t$ satisfies (19) with $a = 2k\pi/n$ and $a = 4k\pi/n$. On the other hand $|f(z)| = 1$ whenever $|z| = 1$. Consequently $f|\{z\colon |z| = 1\}$ satisfies (20) with $a = 4k\pi/n$ and, by Lemma 1 (i), f is a solution of (20) with $a = 4k\pi/n$ as well.

Finally suppose that a homeomorphism $f\colon \Delta_t^* \to \Delta_{t'}^*$ is a solution of (20) with $a = 4k\pi/n$. Since k and n are relatively prime, there exists a pair of integers k_0 and n_0 such that $k_0 k + n_0 n = 1$, i.e. $k_0 k/n = -n_0 + 1/n$. If k_0 is odd the assertion is obvious, so we may assume that k_0 is even. This implies that $f|\Delta_t$ satisfies (19) with $a = 2\pi/n$ and $a = 4\pi/n$. On the other hand $|f(z)| = 1$ whenever $|z| = 1$. Consequently $f|\{z\colon |z| = 1\}$ satisfies (20) with $a = 4\pi/n$ and, by Lemma 1 (i), f is a solution of (20) with $a = 4\pi/n$ as well. This completes the proof.

Now we shall formulate our final result on homeomorphic solutions of (1) and (2), where we consider (9) instead of (2) since every elliptic antigraphy can be written in the form (6).

Theorem 1. *(i) In the elliptic case with $a = a_0$, a_0/π rational $e^{ia_0} \neq 1$, the problem of finding homeomorphic solutions of* (1) *with $a = a_0$ in a natural domain D is equivalent to the problem of finding homeomorphic solutions of* (19) *with $a = 2\pi/n$ in $r_\tau[D]$, where $f = r_\tau \circ g \circ \check{r}_\tau$, $z = r_\tau(s)$ for $s \in D$, τ, $0 < \tau < +\infty$, is chosen so that $r_\tau[\operatorname{fr} D] \supset \{z\colon |z| = 1\}$ for $D \neq \mathscr{E}$, while $\tau = 1$ for $D = \mathscr{E}$, and n, $n \neq 1$, is a positive integer uniquely determined by the requirement for $an/2\pi$ to be an integer, and for n, $an/2\pi$ to be relatively prime. Besides, if a homeomorphism $f\colon \Delta_t \to \Delta_{t'}$ is a solution of* (19) *with $a = 2\pi/n$,* then *f^*, defined by*

$$f^*|\Delta_t = f,\ f^*|\Delta_t^* \setminus \Delta_t = a^* \circ f \circ \check{a}^* | \Delta_t^* \setminus \Delta_t, \tag{21}$$

where $a = 4\pi/n$, is a homeomorphic solution of (20) *with $a = 4\pi/n$, and $f^*[\Delta_t^*] = \Delta_{t'}^*$.*

(ii) In the elliptic case with $a = a_0$, a_0/π rational, $e^{ia_0} \neq 1$, the problem of finding homeomorphic solutions of (9) *with $a = a_0$ in $D \cup a_\tau[D]$, where D is a natural domain, is equivalent to the problem of finding homeomorphic solutions of* (20) *with $a = 4\pi/n$ in $r_\tau[D \cup a_\tau[D]]$, where $f = r_\tau \circ g \circ \check{r}_\tau$, $z = r_\tau(s)$ for $s \in D \cup a_\tau[D]$, and n, $n \neq 1$, is a positive integer uniquely determined by the requirement for $an/4\pi$ to be an integer, and for n, $an/4\pi$ to be relatively prime. Besides, if a homeomorphism $f\colon \Delta_t^* \to \Delta_{t'}^*$ is a solution of* (20) *with $a = 4\pi/n$, then*

$$f_1 = f|\Delta_t,\ f_2 = f|a^*[\Delta_t], \tag{22}$$

where $a = 4\pi/n$, are homeomorphic solutions of (19) *with $a = 4\pi/n$. If, in particular, n is odd, f_1 and f_2 are also solutions of* (19) *with $a = 2\pi/n$. For any n, if, in addition, $|f(z)| = t^*$ whenever $|z| = 1$, then $t^* = 1$, i.e.*

$$|f_1(z)| = |f_2(z)| = 1 \text{ whenever } |z| = 1. \tag{23}$$

Proof. The equivalence of the problem with $a = a_0$ and the problem with $a = 2\pi/n$ in the case of a homography, and $a = 4\pi/n$ in the case of an antigraphy, is a straightforward consequence of Lemmas 2 and 3.

Now, if a homeomorphism $f\colon \Delta_t \to \Delta_{t'}$ is a solution of (19) with $a = 2\pi/n$, then it also satisfies (19) with $a = 4\pi/n$. On the other hand $|f(z)| = 1$ whenever $|z| = 1$. Consequently $f|\{z\colon |z| = 1\}$ satisfies (20) with $a = 4\pi/n$ and, by Lemma 1 (i), f^* is a homeomorphic solution of (20) with $a = 4\pi/n$, such that $f^*[\Delta_t^*] = \Delta_{t'}^*$. This completes the proof of (i).

Finally, if a homeomorphism $f\colon \Delta_t^* \to \Delta_{t'}^*$ is a solution of (20) with $a = 4\pi/n$, then, either directly or by Lemma 1 (ii), we conclude that f_1 and f_2 satisfy (19) with $a = 4\pi/n$. If, in particular, n is odd, then, by Lemma 3 (i), f_1 and f_2 are also solutions of (19) with $a = 2\pi/n$. For any f, if, in addition, $|f(z)| = t^*$ whenever $|z| = 1$, then, either using directly the fact that f is a solution of (20), or applying Lemma 1 (ii), we conclude that $t^* = 1$, i.e. (23) holds. This completes the proof of (ii).

6. Normalized Q-quasiconformal solutions

Now we confine ourselves to Q-quasiconformal solutions of (19) and (20). We remark that now, for $t \neq 0$, t' is restricted by the condition (cf. e.g. [8], p. 40)

(24) $$t^Q \leqslant t' \leqslant t^{1/Q}.$$

According to [12] (pp. 344-345) we introduce the following normalized classes.

Definition 1. $f \in E_Q^{(t,n)}$, where $1 \leqslant Q < +\infty$, $0 \leqslant t < 1$, and n, $n \neq 1$, is a positive integer, iff it is a Q-quasiconformal solution of (19) in the elliptic case with $a = 2\pi/n$, and maps Δ_t onto some $\Delta_{t'}$ so that $f(1) = 1$.

Definition 2. $f \in E_Q^{(n;t)}$, where $1 \leqslant Q < +\infty$, $0 \leqslant t < 1$, and n, $n \neq 1$, is a positive integer, iff $f = f^*|\Delta_t$, where f^* is a Q-quasiconformal solution of (20) in the elliptic case with $a = 4\pi/n$, and maps Δ_t^* onto some $\Delta_{t'}^*$ so that $|f^*(t)| < |f^*(1/t)|$, $f^*(1) = 1$.

Definition 3. $f \in E_Q^{*(n)}$, where $1 \leqslant Q < +\infty$, and n, $n \neq 1$, is a positive integer, iff it is a Q-quasiconformal solution of (19) in the elliptic case with $a = 2\pi/n$, defined on $\mathscr{E}$, and such that $f(0) \neq \infty$, $f(1) = 1$.

Definitions 1–3 imply directly: (a) if $f \in E_Q^{(0,n)} \cup E_Q^{(n;0)}$ then $t' = 0$ and $f(0) = 0$, (b) if $f \in E_Q^{(n;t)}$ then $f[\Delta_t] = \Delta_{t'}$, (c) if $f \in E_Q^{(n;0)}$ then $f^*(\infty) = \infty$, (d) if $f \in E_Q^{*(n)}$ then $f(0) = 0$ and $f(\infty) = \infty$.

It seems natural to ask for relations between the classes $E_Q^{(t,n)}$ and $E_Q^{(n;t)}$. The complete answer is given in the following

Theorem 2. *For n even,*

(25) $$E_Q^{(t,n)} \subset E_Q^{(n;t)} \subset E_Q^{(t,\frac{1}{2}n)},$$

where the indices n and $\frac{1}{2}n$ in the extreme terms cannot be improved for any Q, $1 < Q < +\infty$, and t, $0 \leqslant t < 1$. For n odd,

$$E_Q^{(n;t)} = E_Q^{(t,n)}. \tag{26}$$

Proof. Relations (25) and (26) follow immediately from Theorem 1. In order to show that the indices n and $\frac{1}{2}n$ in the extreme terms of (25) cannot be improved for any Q, $1 < Q < +\infty$, and t, $0 \leqslant t < 1$, we consider functions

$$f(z) = \begin{cases} |z|^{Q^*} \exp i(\arg z + q\sin(\frac{1}{2}n\arg z)) & \text{for } t < |z| \leqslant 1, \\ \lim\limits_{\substack{\zeta\to z \\ |\zeta|>t}} f(\zeta) & \text{for } |z| = t, \end{cases}$$

$$\tilde{f}(z) = \begin{cases} |z|^{\tilde{Q}} \exp i\big(\arg z + \tilde{q}(1-|z|^{\frac{1}{2}n})\sin(\frac{1}{2}n\arg z)\big) & \text{for } t < |z| \leqslant 1, \\ \lim\limits_{\substack{\zeta\to z \\ |\zeta|>t}} \tilde{f}(\zeta) & \text{for } |z| = t, \\ e^{-4\pi n^{-1}i}/\overline{f(e^{4\pi n^{-1}i}/\bar{z})} & \text{for } 1 < |z| \leqslant 1/t, \end{cases}$$

where q, $\tilde{q}$, Q^*, $\tilde{Q}$ are supposed to be positive, and will be specified below. It is clear that f satisfies (19) with $a = 4\pi/n$ but it cannot be continued to a function satisfying (20) with $a = 4\pi/n$, and $\tilde{f}$ satisfies (20) with $a = 4\pi/n$ but $\tilde{f}|\Delta_t$ satisfies (19) with $a = 2\pi/k$ with no $k = \frac{1}{2}n+1$, $\frac{1}{2}n+2, \ldots$ Also $f[\Delta_t] = \Delta_{t'}$, $f(1) = 1$, and $\tilde{f}[\Delta_t^*] = \Delta_{\tilde{t}}^*$, $|\tilde{f}(t)| < |\tilde{f}(1/t)|$, $\tilde{f}(1) = 1$, where $t' = t^{Q^*}$, $\tilde{t}' = t^{\tilde{Q}}$. It remains to choose q, q, Q^*, $\tilde{Q}$ so that f and $\tilde{f}$ be Q-quasiconformal.

First of all we see that f and $\tilde{f}$ are sense-preserving homeomorphisms whenever $q < 2/n$ and $\tilde{q} < 2/n(1-t^{\frac{1}{2}n})$, respectively. Moreover, by (24) we have $Q^{-1} \leqslant Q^* \leqslant Q$ and $Q^{-1} \leqslant \tilde{Q} \leqslant Q$ for $t \neq 0$. Unfortunately these restrictions are necessary but not sufficient, so we have to apply another argument. Obviously $f|\operatorname{int}\Delta_t \setminus \{0\}$ and $\tilde{f}|\operatorname{int}\Delta_t \setminus \{0\}$ are continously differentiable, and

$$\frac{f'_{\bar{z}}(z)}{f'_z(z)} = e^{2i\arg z}\,\frac{Q^* - 1 - \frac{1}{2}nq\cos(\frac{1}{2}n\arg z)}{Q^* + 1 + \frac{1}{2}nq\cos(\frac{1}{2}n\arg z)}, \tag{27}$$

$$\frac{\tilde{f}'_{\bar{z}}(z)}{\tilde{f}'_z(z)} = e^{2i\arg z}\,\frac{\tilde{Q} - 1 - \frac{1}{2}n\tilde{q}\cos(\frac{1}{2}n\arg z) + \frac{1}{2}n\tilde{q}z^{\frac{1}{2}n}}{\tilde{Q} + 1 + \frac{1}{2}n\tilde{q}\cos(\frac{1}{2}n\arg z) - \frac{1}{2}n\tilde{q}z^{\frac{1}{2}n}}. \tag{28}$$

Suppose now that $0 < q \leqslant 2n^{-1}(Q^2-1)/(Q^2+1)$, and calculate the east upper bound of (27) taken over $z \in \operatorname{int}\Delta_t \setminus \{0\}$. It equals either

$$(Q^* - 1 + \tfrac{1}{2}nq)/(Q^* + 1 - \tfrac{1}{2}nq) < 1$$

or

$$-(Q^*-1-\tfrac{1}{2}nq)/(Q^*+1+\tfrac{1}{2}nq)<1.$$

Choosing $Q^* = Q(1-\frac{1}{2}nq)$ in the first case, and $Q^* = Q^{-1}(1+\frac{1}{2}nq)$ in the second, we see that in both cases this bound is exactly $(Q-1)/(Q+1)$. Therefore $f|\operatorname{int}\Delta_t\setminus\{0\}$ is Q-quasiconformal (cf. [8], p. 19) and so is $f|\operatorname{int}\Delta_t$ (cf. [8], p. 43). Applying now the definition of Q-quasiconformality for mappings defined in closed domains bounded by disjoint Jordan curves (cf. Section 2) we see that f is Q-quasiconformal, as desired.

Finally suppose that $0<\tilde{q}\leqslant n^{-1}(Q^2-1)/[Q^2+\frac{1}{2}(1-t^{\frac{1}{2}n})]$, and estimate the least upper bound $M(\tilde{Q},\tilde{q})$ of (28) taken over $z\in\operatorname{int}\Delta_t\setminus\{0\}$. It does not exceed either

$$(\tilde{Q}-1+n\tilde{q})/(\tilde{Q}+1-n\tilde{q})<1$$

or

$$-(\tilde{Q}-1-\tfrac{1}{2}n\tilde{q}+\tfrac{1}{2}n\tilde{q}t^{\frac{1}{2}n})/(\tilde{Q}+1+\tfrac{1}{2}n\tilde{q}-\tfrac{1}{2}n\tilde{q}t^{\frac{1}{2}n})<1.$$

Choosing any pair of $\tilde{Q},\tilde{q}$ so that $M(\tilde{Q},\tilde{q})\leqslant(Q-1)/(Q+1)$ we obtain that $\tilde{f}|\operatorname{int}\Delta_t\setminus\{0\}$ is Q-quasiconformal. Clearly $\tilde{f}|\operatorname{int}(\Delta_t^*\setminus\Delta_t)\setminus\{\infty\}$ is also Q-quasiconformal. Therefore $\tilde{f}|\operatorname{int}\Delta_t^*\setminus\{0,\infty\}$ must be Q-quasiconformal as well (cf. [8], p. 47). Consequently, as in the case of f, we conclude that $\tilde{f}$ is Q-quasiconformal, and this completes the proof.

According to [12] (p. 345) we call mappings of $E_Q^{(t,n)}$ and $E_Q^{(n;t)}$ — *n-cyclic elliptic*, and mappings of $E_Q^{*(n)}$ — *n-cyclic continued elliptic*. As remarked in Section 2, we use the adjective "cyclic" since the set of all homographies h^* with $a = 2k\pi/n$, where n is fixed and k ranges over all integers, forms an n-cyclic group with respect to composition. Mappings of the classes in question may also be called *n-symmetric* since they are a natural extension of the classes of n-symmetric conformal mappings, among others investigated by Littlewood and Paley [9], Basilevich [2, 3], Aleksandrov [1], Jakubowski [7], and Mikołajczyk [13].

7. Characterization of the normalized Q-quasiconformal solutions in terms of complex dilatation

It is essential to characterize the classes in question in terms of complex dilatation.

Theorem 3. (*i*) *In the definitions of* $E_Q^{(t,n)}$ *and* $E_Q^{*(n)}$ *we may replace* (19) *with*

$$\mu(z) = e^{-2i\arg h^{*\prime}(z)}\mu\circ h^*(z) \ \ a.e.\ in\ D, \tag{29}$$

where μ *denotes the complex dilatation of* f, $f(0)=0$ *when* $0\in D$, *and* $f(\infty) = \infty$ *when* $\infty\in D$. *Here* $D=\Delta_t$ *in the case of* $E_Q^{(t,n)}$, *and* $D=\mathscr{E}$ *in the case of* $E_Q^{*(n)}$.

(ii) In the definition of $E_Q^{(n;t)}$ *we may replace* (20) *with*

$$\mu^*(z) = e^{2i\arg a^*(z)}\overline{\mu^*\circ a^*(z)} \text{ a.e. in } \Delta_t^*, \tag{30}$$

where μ^* *denotes the complex dilatation of* f^*, $f^*(0) = 0$ *when* $0 \in \Delta_t^*$, *and* $f^*(\infty) = \infty$ *when* $\infty \in \Delta_t^*$.

Proof. The proof is similar to that given in the case of an analogous result for the class E_Q introduced in [12] (see [12], pp. 312-313). We confine ourselves to the case of $E_Q^{(t,n)}$ since the same method works also for $E_Q^{*(n)}$ and $E_Q^{(n;t)}$.

Definition 1 implies that $f'_z, f'_{\bar z}$ exist a.e. in Δ_t (see e.g. [8], p. 172), and that

$$f'_z(z) = e^{-2\pi i/n} f'_z(e^{2\pi i/n}z) = [f'_\zeta(\zeta)]_{\zeta=e^{2\pi i/n}z},$$
$$f'_{\bar z}(z) = e^{-2\pi i/n} f'_{\bar z}(e^{2\pi i/n}z) = e^{-4\pi i/n}[f'_{\bar\zeta}(\zeta)]_{\zeta=e^{2\pi i/n}z}.$$

Hence (29) follows.

Conversely, suppose that $f\colon \Delta_t \to \Delta_{t'}$ satisfies the conditions given in Theorem 3 (i). By the well known theorem on existence and uniqueness (see e.g. [8], p. 204, in the case where $t = 0$, and [10], p. 26 in the case where $0 < t < 1$) if $f^*\colon \Delta_t \to \Delta_{t'}$ is Q-quasiconformal, $f^*(1) = 1$ (also $f^*(0) = 0$ in the case where $t = 0$), and f^* has μ as its complex dilatation a.e. in Δ_t, then $f^* = f$. On the other hand the mapping $f^{**}\colon \Delta_t \to \Delta_{t'}$, defined by the formula $f^{**}(z) = e^{-2\pi i/n} f(e^{2\pi i/n}z)$ for $z \in \Delta_t$, is also Q-quasiconformal, satisfies $f^{**}(1) = 1$ (also $f^{**}(0) = 0$ in the case where $t = 0$), and its complex dilatation μ^{**} is determined by the formula $\mu^{**}(z) = e^{-4\pi i/n}\mu(e^{2\pi i/n}z)$ a.e. in Δ_t. Since, by (29), $\mu^{**}(z) = \mu(z)$ a.e. in Δ_t, then $f^{**} = f$. Hence f is a solution of (19) and, consequently, $f \in E_Q^{(t,n)}$.

II. The limit case

8. Homeomorphic solutions

In the case under consideration the problems in question can be simplified again.

Lemma 4. *(i) A homeomorphism* $f\colon \Delta_t \to \Delta_{t'}$ *(or* $f\colon \mathscr{E} \to \mathscr{E}$*) is a solution of* (19) *with* $a = a_0$, a_0/π *being irrational, iff it is a solution of* (19) *with any* a, a/π *being irrational.*

(ii) A homeomorphism $f\colon \Delta_t^* \to \Delta_{t'}^*$ *is a solution of* (20) *with* $a = a_0$, a_0/π *being irrational, iff it is a solution of* (20) *with any* a, a/π *being irrational.*

The proof is omitted since it is completely analogous to that given in [12] (pp. 311-312) in the case of Q-quasiconformal solutions $f\colon \Delta_0^* \to \Delta_0^*$ of (20) with a/π irrational, normalized by the conditions $f^*(0) = 0$ and $f^*(1) = 1$ (also by $f^*(\infty) = \infty$, but this is a consequence of (20) and our convention $z/0 = \infty$ for $z \in \mathscr{E}\setminus\{0\}$; cf. Section 1).

Now we shall formulate our final result on homeomorphic solutions of (1) and (2), where we consider (9) instead of (2) since every elliptic homography can be written in the form (6).

Theorem 4. (*i*) *In the elliptic case with $a = a_0$, a_0/π irrational, the problem of finding homeomorphic solutions of* (1) *with $a = a_0$ in a natural domain D is equivalent to the problem of finding homeomorphic solutions of* (19) *with any a, a/π irrational, in $r_\tau[D]$, where $f = r_\tau \circ g \circ \check{r}_\tau$, $z = r_\tau(s)$ for $s \in D$, and τ, $0 < \tau < +\infty$, is chosen so that $r_\tau[\mathrm{fr}\, D] \supset \{z\colon |z| = 1\}$ for $D \neq \mathscr{E}$, while $\tau = 1$ for $D = \mathscr{E}$. Besides, if a homeomorphism $f\colon \Delta_t \to \Delta_{t'}$ is a solution of* (19) *with $a = a_0$, then f^*. defined by* (21) *with $a = 0$, is a homeomorphic solution of* (20) *with any real a, and $f^*[\Delta_t^*] = \Delta_{t'}^*$.*

(*ii*) *In the elliptic case with $a = a_0$, a_0/π irrational, the problem of finding homeomorphic solutions of* (9) *with $a = a_0$ in $D \cup a_\tau[D]$, where D is a natural domain, is equivalent to the problem of finding homeomorphic solutions of* (20) *with any a, a/π irrational, in $r_\tau\big[D \cup a_\tau[D]\big]$, where $f = r_\tau \circ g \circ \check{r}_\tau$ and $z = r_\tau(s)$ for $s \in D \cup a_\tau[D]$. Besides, if a homeomorphism $f\colon \Delta_t^* \to \Delta_{t'}^*$ is a solution of* (20) *with $a = a_0$, then* (22), *where $a = 0$, are homeomorphic solutions of* (19) *with any real a, and* (23) *holds.*

Theorem 4 is a straightforward consequence of Lemmas 2, 4 and 1.

Corollary 1. (*i*) *In the elliptic case with a/π irrational a homeomorphism f is a solution of* (19) *in D iff it satisfies $f(z) = e^{i\arg z} f(|z|)$ for $z \in D\setminus\{0, \infty\}$.*

(*ii*) *In the elliptic case with a/π irrational a homeomorphism f is a solution of* (20) *in $D \cup a^*[D]$ iff it satisfies $f(z) = e^{i\arg z} f(|z|)$ and $f(1/|z|) = 1/\overline{f(|z|)}$ for $z \in D \cup a^*[D]\setminus\{0, \infty\}$.*

Corollary 1 is an easy generalization of two results obtained in [12] (pp. 311-312 and 335-336).

9. Normalized Q-quasiconformal solutions

Now we confine ourselves to Q-quasiconformal solutions of (19) and (20). We remark that now, for $t \neq 0$, t' is restricted by (24). According to [12] (pp. 311, 336 and 344-345) we introduce the following normalized classes.

Definition 4. $f \in E_Q^{(t,\infty)}$, where $1 \leqslant Q < +\infty$ and $0 \leqslant t < 1$, iff it is a Q-quasiconformal solution of (19) in the elliptic case with an a, a/π irrational, defined on Δ_t, and such that $|f(t)| < 1$, $f(1) = 1$.

Definition 5. $f \in E_Q^{(\infty;t)}$, where $1 \leqslant Q < +\infty$ and $0 \leqslant t < 1$, iff $f = f^*|\Delta_t$, where f^* is a Q-quasiconformal solution of (20) in the elliptic case with an a, a/π irrational, defined on Δ_t^*, and such that $|f^*(t)| < |f^*(1/t)|$, $f^*(1) = 1$.

Definition 6. $f \in E_Q^{*(\infty)}$, where $1 \leqslant Q < +\infty$, iff it is a Q-quasiconformal solution of (19) in the elliptic case with an a, a/π irrational, defined on $\mathscr{E}$, and such that $f(0) \neq \infty$, $f(1) = 1$.

Definitions 4–6 imply directly: (a) if $f \in E_Q^{(0,\infty)} \cup E_Q^{(\infty;0)}$ then $t' = 0$ and $f(0) = 0$, (b) if $f \in E_Q^{(\infty;t)}$ then $f[\Delta_t] = \Delta_{t'}$, (c) if $f \in E_Q^{(\infty;0)}$ then $f^*(\infty) = \infty$, (d) if $f \in E_Q^{*(\infty)}$ then $f(0) = 0$ and $f(\infty) = \infty$.

The following analogue of Theorem 2 is an immediate consequence of Theorem 4:

Theorem 5. $E_Q^{(\infty;t)} = \bigcap_n E_Q^{(n;t)} = E_Q^{(t,\infty)} = \bigcap_n E_Q^{(t,n)}$ *and* $E_Q^{*(\infty)} = \bigcap_n E_Q^{*(n)}$.

Some of these relations were established in [12] (p. 345).

Corollary 1 yields (cf. [12], pp. 311-313 and 335-336):

Corollary 2. *In the definitions of* $E_Q^{(t,\infty)}$, $E_Q^{*(\infty)}$ *and* $E_Q^{(\infty;t)}$ *we may replace* (19) *and* (20) *with* $f(z) = e^{i\arg z} f(|z|)$ *for* $z \in D \setminus \{0, \infty\}$, $f(0) = 0$ *when* $0 \in D$, *and* $f(\infty) = \infty$ *when* $\infty \in D$. *Here* $D = \Delta_t$ *for* $f \in E_Q^{(t,\infty)} \cup E_Q^{(\infty;t)}$, *and* $D = \mathscr{E}$ *for* $f \in E_Q^{*(\infty)}$.

According to [12] (p. 345) we call mappings of $E_Q^{(t,\infty)} = E_Q^{(\infty;t)}$ – *limit elliptic*, and mappings of $E_Q^{*(\infty)}$ – *limit continued elliptic*. The adjective "limit" is fully justified by the relations given in Theorem 5. The classes $E_Q = E_Q^{(\infty;0)}$ and $E_Q^* = E_Q^{*(\infty)}$ were studied in detail by Ławrynowicz [12]. On the other hand, E_Q is a subclass of a class introduced by Ławrynowicz in [11] (pp. 161-163).

10. Characterization of the normalized Q-quasiconformal solutions in terms of complex dilatation

It is essential to characterize the classes in question in terms of complex dilatation.

Theorem 6. (*i*) *In the definitions of* $E_Q^{(t,\infty)}$ *and* $E_Q^{*(\infty)}$ *we may replace* (19) *with* (29), *where* μ *denotes the complex dilatation of* f, $f(0) = 0$ *when* $0 \in D$, *and* $f(\infty) = \infty$ *when* $\infty \in D$. *Here* $D = \Delta_t$ *in the case of* $E_Q^{(t,\infty)}$, *and* $D = \mathscr{E}$ *in the case of* $E_Q^{*(\infty)}$.

(*ii*) *In the definition of* $E_Q^{(\infty;t)}$ *we may replace* (20) *with* (30), *where* μ^* *denotes the complex dilatation of* f^*, $f^*(0) = 0$ *when* $0 \in \Delta_t^*$, *and* $f^*(\infty) = \infty$ *when* $\infty \in \Delta_t^*$.

In the case of $E_Q^* = E_Q^{*(\infty)}$ and $E_Q = E_Q^{(\infty;0)}$ this result was obtained in [12] (pp. 335-336 and 311-313). The proof in the general case may be omitted since it is analogous to that given in [12] (pp. 312-313), and to the proof of Theorem 3.

Finally, Theorem 6 and Corollary 2 imply (cf. [12], p. 313 and 336):

Corollary 3. *In the definitions of* $E_Q^{(t,\infty)}$, $E_Q^{*(\infty)}$ *and* $E_Q^{(\infty;t)}$ *we may replace* (19) *and* (20) *with* $\mu(z) = e^{2i\arg z}\mu(|z|)$ *a.e. in* D, *where* μ *denotes the complex dilatation of* f, $f(0) = 0$ *when* $0 \in D$, *and* $f(\infty) = \infty$ *when* $\infty \in D$. *Here* D *has the same meaning as in Corollary* 2.

In conclusion it should be remarked that, by Theorems 3 and 6, all the introduced classes of normalized Q-quasiconformal mappings can be defined with the help of (29) and (30), as suggested in [12] (p. 344).

11. Acknowledgment

The authors are indebted to Professor J. Krzyż for his helpful remarks.

REFERENCES

[1] Александров, И. А., *Граничные значения функционала* $I = I(f, \bar{f}, f', \bar{f}')$ *на классе голоморфных однолистных в круге функций*, Сибирск. Мат. Ж. 4 (1963), стр. 17–31.

[2] Basilewitsch, J., *Zum Koeffizientenproblem der schlichten Funktionen*, Мат. Сб. 1 (1936), pp. 211–228.

[3] Базилевич, И., *Дополнение к работе „Zum Koeffizientenproblem der schlichten Funktionen"* и *„Sur les théorèmes des Koebe-Bieberbach"*, Мат. Сб. 2 (1937), стр. 681–697.

[4] Behnke, H., und Sommer, F., *Theorie der analytischen Funktionen einer komplexen Veränderlichen*, 3. Aufl., Springer-Verlag, Berlin-Heidelberg-New York 1965.

[5] Chądzyński, J., and Ławrynowicz, J., *On Quasi-conformal Solutions of a Class of Functional Equations*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), to appear.

[6] — *On Homeomorphic Solutions of a Class of Functional Equations*. ibid. 18 (1970), to appear.

[7] Jakubowski, Z. J., *Les fonctions univalentes, p-symétriques et bornées dans le cercle unité*, Ann. Polon. Math. 20 (1968). pp. 119–148.

[8] Lehto, O., und Virtanen, K. I., *Quasikonforme Abbildungen*, Springer-Verlag, Berlin-Heidelberg-New York (1965).

[9] Littlewood, J. E., and Paley, R.E.A.C., *A Proof that an Odd Schlicht Function has Bounded Coefficients*, J. London. Math. Soc. 7 (1932), pp. 167–169.

[10] Ławrynowicz, J., *On the Parametrization of Quasiconformal Mappings in an Annulus*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 18 (1964), pp. 23–52.

[11] — *On certain Functional Equations for Quasiconformal Mappings*, Ann. Polon. Math. 20 (1968), pp. 153–165.

[12] — *On a Class of Quasiconformal Mappings with Invariant Boundary Points I-II*, ibid. 21 (1969), pp. 309–324 and 325–347.

[13] Mikołajczyk, L., *A Theorem on Distortion for Univalent p-symmetrical Functions Bounded in the Circle* $|z| > 1$, Prace Mat. 12 (1968), pp. 35–51.

[14] Saks, S., and Zygmund, A., *Analytic Functions*, 2nd ed., PWN-Polish Scientific Publishers, Warszawa 1965.

[15] Schwerdtfeger, H., *Geometry of Complex Numbers*, Oliver and Boyd, Edinburgh-London 1962.

Streszczenie

Autorzy zajmują się rozwiązaniami homeomorficznymi równań funkcyjnych postaci $g\circ h = h\circ g$ oraz $g\circ a = a\circ g$, gdzie h jest dowolną ustaloną homografią eliptyczną, zaś a – dowolną ustaloną antygrafią eliptyczną (antygrafia jest to złożenie odbicia względem osi rzeczywistej z homografią). Zagadnienie to wiąże się z grupami cyklicznymi homografii i antygrafii. W szczególności autorzy rozpatrują rozwiązania Q-quasikonforemne tych równań, wprowadzają pewne znormalizowane klasy rozwiązań Q-quasi-konforemnych, uzyskują dla nich pewne warunki konieczne i dostateczne oraz badają związki między tymi klasami. Wprowadzone klasy dają m.in. naturalne rozszerzenie znanych klas odwzorowań konforemnych n-symetrycznych.

Резюме

Авторы занимаются гомеоморфными решениями функциональных уравнений вида $g\circ h = h\circ g$ и $g\circ a = a\circ g$, где h-произвольная фиксированная гомография (т.е. дробно-линейное преобразование) эллиптического типа, тогда как a — произвольная фиксированная антиграфия (т.е. суперпозиция симметрии относительно действительной оси координат и гомографии) эллиптического типа. Проблема эта связана с циклическими группами гомографии и антиграфии. В частности авторы рассматривают Q-квазиконформные решения этих уравнений, вводят некоторые нормализированные классы Q-квазиконформных решений, получают для них несколько необходимых и достаточных условий, а также изучают соотношения между этими классами. Введенные классы дают естественное расширение известных классов n-симметрических конформных отображений.