ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. XIX, 5 SECTIO A 1965

Z Katedry Matematyki Wydziału Ekonomicznego UMCS
Kierownik: doc. dr Zdzisław Lewandowski

ZDZISŁAW LEWANDOWSKI

On some Problems of M. Biernacki Concerning Subordinate Functions and on some Related Topics

O pewnych zagadnieniach M. Biernackiego dotyczących podporządkowania funkcji i pewnych problemach pokrewnych

О некоторых задачах М. Бернацкого, касающихся подчинения функций и других аналогичных задачах

1. Introduction

In this paper we are concerned with the notions of subordination and domination which will be now defined. Suppose $C_r = \{z: |z| < r\}$ where $r > 0$ and suppose f, F are functions regular in C_1 which satisfy: $f(0) = F(0) = 0$. We say that f is subordinate to F in C_r and denote this (f, F, r) if there exists a function w regular in C_r, $w(0) = 0$, $|w(z)| \leqslant r$ in C_r, such that $f(z) = F(w(z))$ in C_r. If F is univalent, (f, F, r) means that the image of C_r under f is contained in the image of C_r under F. F will be called domain majorant of f in C_r.

On the other hand, if $|f(z)| \leqslant |F(z)|$ holds in C_r with $r \in (0, 1\rangle$, then we say that f is dominated by F in C_r and denote this in the following manner: $|f, F, r|$. F will be called a dominant of f in C_r. In the particular case $F(z) = z$ both relations (f, F, r), $|f, F, r|$ are equivalent by the Schwarz Lemma, in general case they are not, however, equivalent. Moreover, by the Schwarz Lemma and the maximum principle it follows that $(f, F, r) \Rightarrow (f, F, r')$, $|f, F, r| \Rightarrow |f, F, r'|$ for any $r' \in (0, r)$. It was M. Biernacki who was concerned as the first with the relation between subordination and domination [7], [8]. He obtained for some classes of regular functions f, F with univalent F the theorems of the following type: $(f, F, 1) \Rightarrow |f, F, r|$ where $r \in (0, 1)$ does not depend on the particular choice of functions involved. Some results of Biernacki were generalized and extended by G. M. Golusin [10] and Shah Tao-shing [25]. The converse relation was investigated by the present author in [14], further results in this direction were obtained in [15] and [16], as well as in a series of papers written in common by the present author and A. Bielecki [1]-[5] and concerning also the problem first considered by Biernacki. We have given a simple geometrical method which showed to be very convenient in investigating some classes of regular functions. In Biernacki's papers we meet two kinds of assumptions. First he assumed that both the subordinate function and its majorant belong to the same class (e.g. both are univalent, resp. both are starlike). Later on he put on the subordinate functions less restrictive assumptions than those involving majorants. In what follows we shall start with the assumptions analogous to the latter ones. Some of the results presented here were already published in various journals whereas some are new. Although the circle of problems presented here is by no means exhausted, the present author believes that it is worthwhile to give a systematic and complete treatment on the subject in order to exhibit the methods applied here. However, a complete proof will be given only in case the corresponding theorem is published here for the first time. The last chapter contains a set of related problems which still remain unsolved, resp. only a partial solution of them is known.

2. Notations. Preliminary results

Let S be the class of functions $F(z) = z + a_2 z^2 + \ldots$ regular and univalent in C_1. In what follows we shall be concerned with the following subclasses of S: the class $\bar{S}$ of functions with real coefficients; the class S_c of convex functions; the class S_a of a-starlike functions, i.e. the class of functions $f \epsilon S$ such that $\operatorname{re}\{zf'(z)/f(z)\} > a$ with $a \epsilon \langle 0, 1)$. The case $a = 0$ corresponds to the class S_0 of functions starlike w.r.t. the origin.

Let O_r^n be the domain whose boundary consists of the left half of the circumference $|z| = r^{n+1}$ and of two circular arcs through $z = r^n$ tangent to $|z| = r^{n+1}$ at $z = \pm i r^{n+1}$ ($n \geqslant 0$ is an integer). Suppose K is a fixed subclass of S and $D(K, r)$ is the region of variability of the expression $g(z_1)/g(z_2)$, where z_1, z_2 range over the circumference ∂C_r and g ranges over the class K. Using this notation we can now state a theorem first published in [17] which represents a general solution to the converse of Biernacki's problem.

Theorem 1. *Suppose $r_0 \epsilon (0, 1)$ and $K \subset S$. Then $|f, F, 1|$ implies (f, F, r_0) for every $F \epsilon K$ and every $f(z) = a^n z^n + a_{n+1} z^{n+1} + \ldots$, regular in C_1 and such that $a_n \geqslant 0$ if and only if either for any $r \epsilon (0, r_0)$ the sets $D(K, r)$, $\bar{O}_r^{n-1}$ are disjoint $(n > 1)$, or have only one point $z = 1$ in common $(n = 1)$.*

3. The second problem of Biernacki

We now consider a fixed subclass K of S and we want to determine the greatest possible real number $r_0 \in (0, 1)$ such that for any f regular in C_1 and such that $f(0) = 0, f'(0) \geqslant 0$ the assumptions $(f, F, 1)$, $F \in K$, imply $|f, F, r_0|$. In [7] Biernacki proved the following

Theorem 2. *There exists a number $r_0 \geqslant 1/4$ such that for any $F \in S$ and any function f regular in C_1 and such that $f(0) = 0, f'(0) \geqslant 0$ the condition $(f, F, 1)$ implies $|f, F, r_0|$ and no greater number has this property for all admissible f, F.*

In 1951 Golusin proved that $\frac{1}{3} < r_0 \leqslant \frac{1}{2}(3-\sqrt{5})$, [10]. He also proved that for $F(z) = z(1+z)^{-2}$ and $f(z) = F(z^2)$ the disk where f is dominated by F has radius $\frac{1}{2}(3-\sqrt{5})$.

A few years later Shah Tao-shing proved [25] that $r_0 = \frac{1}{2}(3-\sqrt{5})$. In connection with the second problem of Biernacki Golusin was concerned with the majorants $F \subset S_0$ and obtained the following

Theorem 3. *If $F \in S_0$ and f is regular in C_1 and satisfies $f(0) = 0$, $f'(0) \geqslant 0$, then $(f, F, 1)$ implies $|f, F, r_0|$ with $r_0 = \frac{1}{2}(3-\sqrt{5})$; r_0 cannot be replaced by any greater number.*

4. The second converse problem

The second problem of Biernacki concerns the estimate of the radius of domination for a given class of majorants and a given class of subordinate functions.

We can put in a natural way an analogous question so far as the converse problem is concerned, i.e. given a class of dominants and given a class of dominated functions find the greatest number R such that $|f, F, 1|$ implies (f, F, R) for all admissible functions f, F. Taking S as the class of dominants and assuming that the dominated functions f are regular in C_1 and satisfy $f(0) = 0, f'(0) \geqslant 0$, we realize that the methods leading to Theorem 2 and its generalizations cannot be used for the converse problem. However, we proved [14] the following.

Theorem 4. *There exists a number R, $0.21 < R \leqslant R_0 = 0.29\ldots$, such that for any $F \in S$ and any f regular in C_1 with $f(0) = 0, f'(0) > 0$, the assumption $|f, F, 1|$ implies (f, F, R).*

Here R_0 is the unique positive root of the equation $x^3+x^2+3x-1 = 0$. If the admissible dominants F range over S_0 then we have, cf. [15], the following.

Theorem 5. *If $F \in S_0$ and f is regular in C_1 and satisfies: $f(0) = 0$, $f'(0) > 0$, then $|f, F, 1|$ implies (f, F, R_0) where $R_0 = 0.29\ldots$ is the number defined in Theorem 4.*

If $F_1(z) = z(1+z)^{-2}$ and $f_1(z) = zF_1(z)$, then the greatest disk of subordination has radius R_0.

It is an obvious consequence of Theorem 5 that the constant R defined in Theorem 4 does not exceed R_0. In [17] we gave the following generalization of Theorem 5.

Theorem 6. *Let R_{n-1}, $n = 1, 2, \ldots$ denote the smallest positive root of the equation $x^n = (1-x)^2(1+x)^{-2}$. If $F \in S_0$ and $f(z) = a_n z^n + \ldots$ is regular in C_1, $a_n \geqslant 0$, then $|f, F, 1|$ implies (f, F, R_{n-1}). In case $F_1(z) = z(1+z)^{-2}$ and $f_n(z) = (-1)^{n+1} z^n F(z)$ the number R_{n-1} cannot be replaced by any greater number.*

Theorem 1 plays an essential role in proving Theorem 6. Moreover, Theorem 5 is a particular case of Theorem 6 ($n = 1$).

5. Coefficients of dominated functions

It is possible to give estimates of Taylor coefficients of subordinate functions in terms of coefficients of the majorant, cf. e.g. [11], p. 406-409.

J. E. Littlewood, cf. [19], p. 222, proved that if $f(z) = a_1 z + a_2 z^2 + \ldots$ is regular in C_1 and $(f, F, 1)$ holds with $F \in \bar{S}$, then $|a_n| \leqslant n$.

It is quite natural to ask whether the assumption $|f, F, 1|$ leads to analogous estimates. Let G_n be Landau numbers, cf. [13], p. 29, defined as follows:

$$G_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \ldots + \left(\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n}\right)^2, \quad G_0 = 1.$$

Let N be the class of functions $w(z)$ regular in C_1 and such that $|w(z)| \leqslant 1$ in C_1.

If $w(z) = b_0 + b_1 z + \ldots$ and $w(z) \in N$, then $|b_0 + b_1 + \ldots + b_n| \leqslant G_n$, $n = 0, 1, 2, \ldots$, cf. [13], p. 26. These estimates are sharp. Using this results we shall prove the following.

Theorem 7. *If $F \in \bar{S}$, $f(z) = a_1 z + a_2 z^2 + \ldots$ is regular in C_1 and $|f, F, 1|$ holds, then $|a_n| \leqslant nG_{n-1}$.*

Proof. We may suppose without loss on generality that $a_1 \geqslant 0$. There exists $w(z) = b_0 + b_1 z + \ldots \in N$ $b_0 \in \langle 0, 1\rangle$ such that $f(z) = F(z)w(z)$. If $F(z) = z + A_2 z^2 + \ldots$ then the coefficients a_n can be expressed in the following way:

$$a_n = \sum_{k=0}^{n-1} b_k A_{n-k}, \quad n = 1, 2, \ldots \tag{1}$$

Put $F(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta})$. Since A_n are real, we have $v(re^{i\theta}) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta$. The function $v(re^{i\theta})$ has a constant sign in $(0, \pi)$. After a multiplication by $\sin n\theta$ and integration we obtain

$$(2) \qquad A_n r^n = \frac{2}{\pi}\int_0^{\pi} v(re^{i\theta})\sin n\theta\, d\theta, \quad 0 < r < 1.$$

Hence $a_n = \frac{2}{\pi}\int_0^{\pi} v(re^{i\theta}) \sum_{k=0}^{n-1} b_k r^{k-n}\sin[(n-k)\theta]\, d\theta$, i.e.

$$(3) \qquad a_n = \frac{2}{\pi}\int_0^{\pi} v(re^{i\theta}) S_{n-1}(\theta)\, d\theta,$$

where

$$S_{n-1}(\theta) = \sum_{k=0}^{n-1} b_k r^{k-n}\sin[(n-k)\theta].$$

Put $P^{n-1}(z) = \sum_{k=0}^{n-1} b_k z^{n-k}$. Hence $z^n P_{n-1}\left(\frac{1}{z}\right) = b_0 + b_1 z + \ldots + b_{n-1} z^{n-1}$. Now, $|S_{n-1}(\theta)| = \frac{1}{2}\left|P_{n-1}\left(\frac{1}{\bar z}\right) - P_{n-1}\left(\frac{1}{z}\right)\right|$, $z = re^{i\theta}$, and Landau's estimate quoted above gives $|S_{n-1}(\theta)| \leqslant G_{n-1} r^{-n}$. It is well known, cf. [15], p. 221 that if $T_n(\theta) = c_1 \sin\theta + \ldots + c_n \sin n\theta$ and $|T_n(\theta)| \leqslant 1$ for $\theta \in \langle 0, 2\pi\rangle$, then $|T_n(\theta)| \leqslant n|\sin\theta|$. With $T_{n-1}(\theta) = r^n S_{n-1}(\theta)/G_{n-1}$ we obtain the inequality:

$$|S_{n-1}(\theta)| \leqslant G_{n-1} r^{-n} n |\sin\theta|.$$

Using this and the formulas (2), (3) we have

$$|a_n| \leqslant \frac{2}{\pi} n r^{-n} G_{n-1}\int_0^{\pi} v(re^{i\theta})\sin\theta\, d\theta = n r^{1-n} G_{n-1}$$

and in the limiting case $r \to 1$ we finally obtain the inequality $|a_n| \leqslant nG_{n-1}$.

The bound obtained is not sharp, e.g. for $n = 2$ we have according to Theorem 7: $|a_2| \leqslant 2(1+\frac{1}{4})$, whereas in fact we can derive the sharp estimate $|a_2| \leqslant 2$ as a corollary of Theorem 9. It is well known that $G_n \sim \frac{1}{\pi}\log n$, hence according to Theorem 7 we have $a_n = O(n\log n)$.

In a recent paper [21] T. Mac Gregor proved a sharper result: $|a_n| \leqslant n$.

The two following theorems are further examples of analogues between subordination and domination in absolute value [16]:

Theorem 8. *If $f(z) = a_1z+\ldots$ and $F(z) = A_1z+\ldots$ are analytic in C_1 and $|f, F, 1|$ holds then*

$$\sum_{k=1}^{n} |a_k|^2 \leqslant \sum_{k=1}^{n} |A_k|^2, \quad n = 1, 2, \ldots$$

A corresponding theorem with the assumption $(f, F, 1)$ can be found in [19], p. 168, also cf. [11], p. 406. For the second coefficient a_2 a better estimate than that given by Theorem 7 can be obtained.

Theorem 9. *Under the assumptions of Theorem 8 we have*

$$|a_2| \leqslant 1+|A_2|^2/4|A_1|, \quad \text{if} \quad A_1 \neq 0;$$
$$|a_2| \leqslant |A_2|, \quad \text{if} \quad A_1 = 0.$$

6. Some problems involving the derivative

In [7] another problem was also investigated by Biernacki. Given a function $f(z) = a_1z+a_2z^2+\ldots$, $a_1 \geqslant 0$, regular in C_1 which satisfies $(f, F, 1)$ with $F \in S$. Does there exist an absolute constant $r_1 > 0$ such that $|f', F', r_1|$ holds. Biernacki found an affirmative answer under a supplementary assumption of univalence of f. He could prove that there exists $r_1 > 0.1$ such that $(f, F, 1)$ with univalent f and $F \in S$ implies $|f', F', r_1|$. Golusin improved this result [11] and showed that $r_1 > 0.12$ even if f is not univalent. A few years ago Shah Tao-shing found the best possible $r_1 = 3-\sqrt{8} = 0.17\ldots$, [26].

In [16] we have stated and proved similar theorems where the assumption of subordination is replaced by the assumption of modular domination.

Theorem 10. *If $f(z) = az+\ldots$, $0 \leqslant a < 1$, is regular in C_1 and $F \in S$, then $|f, F, 1|$ implies $|f', F', s(a)|$, where $s(a)$ is the smallest positive root of the equation $ax^3-3ax^2-3x+1 = 0$. The quantity $s(a)$ cannot be replaced by any greater number.*

If $F_o(z) = z(1+z)^{-2}$ and $f_o(z) = (1+az)^{-1}(z+a)F_o(z)$, then $x > s(a)$ implies $f_o'(x) > F_o'(x)$. This shows that $s(a)$ is best possible even in the case F ranges over the more restrictive class S_o. It is easy to see that $s(a)$ is strictly decreasing in $\langle 0, 1)$ and $\lim_{a\to 1-} s(a) = 2-\sqrt{3}$. This implies

Theorem 11. *If f is regular in C_1, $f(0) = 0, f'(0) \geqslant 0$ and $F \in S$, then $|f, F, 1|$ implies $|f', F', 2-\sqrt{3}|$ and $2-\sqrt{3}$ cannot be replaced by any greater number.*

If the class of majorants is restricted to S_o, neither Theorem 10, nor Theorem 11 can be sharpened. Even in case the dominated function f is univalent and starlike the radius of domination remains the same. In fact we have proved [16] the following

Theorem 12. *If $F \in S_0$, f is univalent and starlike in C_1 and satisfies $f(0) = 0, f'(0) \geq 0$, then $|f, F, 1|$ implies $|f', F', 2-\sqrt{3}|$ and the constant $2-\sqrt{3}$ is best possible.*

We can also state an analogue of a result due to Schiffer [24] where the assumption $(f, F, 1)$ is replaced by $|f, F, 1|$, cf. [16].

Theorem 13. *If f is regular in $C_1, f(0) = 0$, and $F \in S$, then $|f, F, 1|$ implies*

$$|f'(z)| \leq (1+|z|)(1-|z|)^{-3}$$

Equality holds for $F_0(z) = z(1-z)^{-2}$ and $f_0(z) = F_0(z)$.

The estimate of $|f'|$ is the same as given by Schiffer.

7. First problem of Biernacki and its converse.

The first problem of Biernacki which was stated by him earlier and is less difficult is obtained as a particular case of the second problem by making an additional supposition that f which is subordinate to F is of the same type as F. In other words, if $F \in K \subset S$, then $f'(0) > 0$ and $f(z)/f'(0) \in K$.

Biernacki proved [7], [8], by use of Julia's variational method following theorems.

Theorem 14. *If $F \in S, f'(0) > 0, (f'(0))^{-1} f \in S$ then $(f, F, 1)$ implies $|f, F, r^0|$, where $r^0 = 0.39\ldots$ is the unique positive root of the equation $2\ln(1+x)/(1-x) - 4\arctan x = \pi$. The number r^0 cannot be replaced by any greater number.*

Theorem 15. *If $f'(0) > 0, (f'(0))^{-1} f \in S_0$ and $F \in S_0$ then $(f, F, 1)$ implies $|f, F, r_g^0|$, where $r_g^0 = \sqrt{2}-1$ cannot be replaced by any greater number.*

Theorem 16. *If $f'(0) > 0, (f'(0))^{-1} f \in S_c$ and $F \in S_c$ then $(f, F, 1)$ implies $|f, F, r_c^0|$ where $r_c^0 = 0.543\ldots$ which is the positive root of the equation $2\arcsin x + 4\arctan x = \pi$ cannot be replaced by any greater number.*

The Theorems 14-16 due to Biernacki were generalized in a common paper by A. Bielecki and the present author [2]. It was proved that the constants r^0, r_g^0, r_c^0 remain unchanged even in case f is regular in C_1 and satisfies $f'(0) > 0, f(z) \neq 0$ for $z \neq 0$.

It seems to be natural to consider the converse of both the first and second problem of Biernacki. The research in this direction done by A. Bielecki and the present author results in proving some new theorems by a quite useful method based on the notion of homotopy. This method is described in the next chapter.

8. Homotopic subordination and domination chains

Let H be the class of functions $h(z, t) = a_1(t)z + a_2(t)z^2 + \ldots t \in \langle t_1, t_2 \rangle$, $a_1(t) > 0$, which satisfy the following conditions:

(i) for any fixed $t \in \langle t_1, t_2 \rangle$ the function $h(z, t)$ is regular and univalent in C_1,

(ii) for any fixed $z \in C_1$ the function $h(z, t)$ and its derivative $h'_t(z, t)$ are continuous w.r.t. $t \in \langle t_1, t_2 \rangle$.

We say that $h(z, t) \in H$ is increasing in absolute value inside C_r, $0 < r \leqslant 1$ if $t', t'' \in \langle t_1, t_2 \rangle$, $t' < t''$ implies $|h(z, t'), h(z, t''), r|$. Similarly, we say that $h(z, t) \in H$ is domainwise increasing inside C_r, $0 < r \leqslant 1$, if t', $t'' \in \langle t_1, t_2 \rangle$, $t' < t''$ implies $(h(z, t'), h(z, t''), r)$.

Let H^a be the subclass of H consisting of all $h(z, t)$ which satisfy $\varphi \in S_a$ with $\varphi(z, t) = h(z, t)/h'_z(0, t)$.

In [1], p. 47-49 A. Bielecki and the present autor have proved the following results which enabled them to create a general method of solving both the first and the second problem of Biernacki, as well as their converse.

Lemma 1. *If $h \in H$ and either $|\arg\{h'_t(z, t)/zh'_z(z, t)\}| < \pi/2$, or $h'_t(z, t) = 0$ for any $z \in C_r$, $t \in \langle t_1, t_2 \rangle$ then h is domainwise increasing in C_r. Conversely, if $h \in H$ is domainwise increasing in C_r, then either $|\arg\{h'_t(z, t)/zh'_z(z, t)\}| < \pi/2$ (or $h'_t(z, t) = 0$ for $t \in \langle t_1, t_2 \rangle$).*

Lemma 2. *The function $h \in H$ is increasing in absolute value in C_r if and only if either $|\arg\{h'_t(z, t)/h(z, t)\}| < \pi/2$, or $h'_t(z, t) = 0$ for $t \in \langle t_1, t_2 \rangle$ and $z \in C_r$.*

Homotopies $h(z, t)$ which are domainwise increasing resp. increasing in absolute value were considered already by many authors, Löwner, Schaeffer-Spencer, Pommerenke, cf. ([20], [23], [22].

The lemma 1 has been also applied by Bielecki and the present author in [6] where a simple proof of the following theorems was given: the class of close-to-convex functions (introduced by W. Kaplan in [12]) is a subclass of linearly accessible functions (introduced by M. Biernacki in [9]).

This proof was considerably simpler than that earlier published by the present author [18].

Let $R(a)$ be the smallest positive root of the equation

$$\arcsin \frac{2r}{1 + r^2 + \dfrac{a}{1-a}(1-r)} + 2\arctan r = \pi/2, \qquad a \in \langle 0, 1).$$

It is easy to see that $R(a) \in (0, 1)$.

Using the Lemmas 1,2 Bielecki and the present author obtained in [1] the following basic results for the first problem of Biernacki and its converse.

Theorem 17. *If* $a \in \langle 0, 1)$, $h(z, t) \in H^a$ *for any* $t \in \langle t_1, t_2\rangle$ *and* $h(z, t)$ *increases in* C_1 *in absolute value, then also* $h(z, t)$ *increases domainwise in* $C_{R(a)}$.

Theorem 18. *If* $a \in \langle 0, 1)$, $h(z, t) \in H^a$ *for any* $t \in \langle t_1, t_2\rangle$ *and* $h(z, t)$ *increases in* C_1 *domainwise, then also* $h(z, t)$ *increases in absolute value in* $C_{R(a)}$.

By the above stated results both Biernacki's problem and its converse are reduced to the investigation of a relevant homotopy $h(z, t)$. The following homotopy

$$h(z, t) = \begin{bmatrix} 0 & \text{for} \quad z = 0 \\ z\left[\dfrac{f(z)}{z}\right]^{1-t}\left[\dfrac{F(z)}{z}\right]^{t}, & z \in C_1,\ 0 \leqslant t < 1, \end{bmatrix}$$

shows to be quite useful in some cases. It is easy to see that this homotopy runs over H^a if $F \in S_a$ and $f(z)/f'(0) \in S_a\,(f^e(0) > 0)$. Using this homotopy as well as Theorem 17 we can obtain the solution of the converse of first problem of Biernacki. Putting $a = 0$, resp. $a = \frac{1}{2}$ we obtain for $R(a)$ the same values as those given by Theorems 15 and 16.

9. The class S_v and some generalizations of the first problem of Biernacki and its converse

Let $v(t)$, $t \in \langle 0, 1\rangle$, be a real, non-decreasing and lower semicontinuous function vanishing at $t = 0$ and put

(A) $$r(v) = \sup(x : 0 \leqslant x \leqslant 1,\ v(x) + 2\arctan x < \pi/2);$$

this implies that $0 \leqslant r(v) < 1$. Let S_v be the class of all functions $F(z) = z + A_2 z^2 + \ldots$ regular in C_1 and such that for any $r \in \langle 0, 1)$ we have

$$\left|\arg \frac{zF'(z)}{F(z)}\right| \leqslant v(r), \quad |z| \leqslant r < 1.$$

The present author and A. Bielecki have proved [2] the following.

Theorem 19. *If* $F \in S_v$ *and* $f(z) = a_n z^n + \ldots$, $a_n > 0$, *is regular in* C_1 *and* $f(z) \neq 0$ *for any* $0 < |z| < 1$ *then* $(f, F, 1)$ *implies* $|f, F(z^n), r(v)|$ *where* $r(v)$ *is defined by* (A).

Theorems 14-16 due to Biernacki show to be corollaries of Theorem 19. We only need to take $n = 1$ and $v(r)$ equal $\ln(1+r)/(1-r)$, $\arcsin \times$

$\times \dfrac{2r}{1+r^2+\dfrac{a}{1-a}(1-r^2)}$, $\arcsin r$ resp. which are the corresponding bounds for the classes S, S_a, S_c, cf. e.g. [11], p. 146, [1], p. 46 and [27].

The essential tool in proving Theorem 19 is Lemma 1. Moreover, it is worthwhile to mention that the assumptions concerning $f(z)$ in Theorem 19 are less restrictive than those made by Biernacki. An analogous result was also given by Bielecki and the present author for the converse problem, cf. e.g. [3], p. 300.

Theorem 20. *If* $r(v) > 0, f'(0) > 0, f(z)/f'(0) \in S_v$ *and* $F \in S_v$ *then* $|f, F, 1|$ *implies* $(f, F, r(v))$ *where* $r(v)$ *is given by* (Λ).

Using this as well as the well known estimate of $\arg\{zf'(z)/f(z)\}$ for the class S we obtain, cf. [3], p. 301, the following

Theorem 21. *If* $F \in S, f'(0) > 0, f(z)/f'(0) \in S$ *then* $|f, F, 1|$ *implies* (f, F, r^0), *where* $r^0 = 0.39\ldots$ *is the Biernacki constant defined in Theorem* 14; r^0 *cannot be replaced by any greater number.*

This theorem gives the solution to the converse of the first problem of Biernacki as stated in Theorem 14.

In [4], [5] some general theorems due to Bielecki and the present author have been proved under weaker assumptions:

Theorem 22. *Suppose* $F(z)$ *and* $w(z) = a_p z^p + \ldots\ p \geqslant 1, a_p \neq 0$, *are regular in* C_1, $F(0) = 0$, $0 < |w(z)| < 1$ *in* $0 < |z| < 1$, $w(z) \not\equiv e^{i\alpha} z$ *with real* α. *Suppose, moreover,* $F \in S_v$ *and* $f(z) = F(w(z))$. *Then the relation* $|f, F, \varrho|$ *holds with* ϱ *being the unique root of the equation* $v(r) + \arctan\varphi(r) = \pi/2$, *where* φ *is a (rather complicated) explicit expression depending on* r, $\arg a_p$ *and* p *and quoted in* [4], p. 92. *For* $p = 1$, $\arg a_p = 0$, *we obtain in particular the results concerning the first problem of Biernacki.*

Theorem 23. *Suppose* $F \in S_v$ *and* $g(z) = b_0 + b_1 z + \ldots$ *is regular in* C_1 *and such that* $0 < |g(z)| < 1$. *If* $f(z) = g(z)F(z)$, *then* (f, F, ϱ) *holds with* ϱ *being the least root of the equation* $v(r) + \arctan\varphi(r) = \dfrac{\pi}{2}$.

Here φ depends on b_0 in a rather complicated way. The case $\arg b_0 = 0$ corresponds to Theorem 20.

10. Estimates of radius of subordination in some problems involving the class S_v^0.

Theorem 21 with an explicit function $v(r)$ corresponding to the class S gives a sharp estimate.

On the other hand, the problem whether the estimate of radius of subordination given by Theorem 20 is sharp, still remains open. In this chapter we show that if the class considered and the corresponding function $v(r)$ are chosen in a suitable manner then the bound obtained by Theorem 20 is sharp.

Let S_v^0 be the class of functions $F(z) = z+A_2z^2+\ldots$ regular in C_1 such that $F'(z) \neq 0$ in C_1 and $|F(z)| > 0$ for $0 < |z| < 1$. Suppose, moreover, that

$$v_0(r) = \sup_{F\epsilon S_v^0}\left(\sup_{|z|\leqslant r}\arg\frac{zF'(z)}{F(z)}\right)$$

is continuous in $\langle 0, 1)$. We may assume without loss on generality that S_v^0 contains $F(z) \not\equiv z$. Then $\arg\dfrac{zF'(z)}{F(z)}$ is a non-constant harmonic function in C_1. This implies that $\sup\limits_{|z|\leqslant r}\arg\dfrac{zF'(z)}{F(z)}$ and also $v(r)$ are strictly increasing functions of $r\epsilon\langle 0, 1)$. We can take

$$v_0(r) = \sup_{F\epsilon S_v^0}\left[\sup_{|z|=r}\arg\frac{zF'(z)}{F(z)}\right].$$

With this choice of $v(r)$ we see that for any $\varepsilon > 0$ there exists $F\epsilon S_v^0$ and $z = re^{i\theta}$ such that $\arg\dfrac{zF'(z)}{F(z)} = v(r)-\varepsilon$. We now prove

Theorem 24. *$F\epsilon S_v^0, f'(0) > 0, f(z)/f'(0)\epsilon S_v^0$ then $|f, F, 1|$ implies $(f, F, r(v_0))$ where $r(v_0)$ is given by the formula (A). The number $r(v_0)$ cannot be replaced by any greater number.*

Proof. The first part of this theorem follows from Theorem 20. We now prove the second part. Suppose that with the assumptions of Theorem 24 the relation (f, F, R) holds for some $R > r(v_0)$. Given $\varepsilon > 0$ choose real x and y so that $r(v_0) < x < y <$ R and also

$$\pi/2 < v_0(x)-\varepsilon+2\arctan x < \pi, \tag{B}$$

where $v_0(y) < \pi/2$. Such a choice is possible by continuity of $v_0(r)$. Suppose $F\epsilon S_v^0$ is such that for some u, $|u| = x$, we have

$$\arg\frac{uF'(u)}{F(u)} = v_0(x)-\varepsilon \tag{C}$$

There exists a real θ such that

$$\arg\frac{1+ue^{i\theta}}{1-ue^{i\theta}} = 2\arctan x \tag{D}$$

Put for $|z| \leqslant 1$ and $t \epsilon (0, 1)$

$$\text{(E)} \quad \begin{cases} g(z,t) = F(pz)h(z,t) \\ h(z,t) = 1-(1-t)(1+tpze^{i\theta})/(1-tpze^{i\theta}), \\ p = \dfrac{x}{y},\ p_0 = \dfrac{u}{p} \end{cases}$$

From the equation $\dfrac{zg_z'(z,t)}{g(z,t)} = \dfrac{pzF'(pz)}{F(pz)} + \dfrac{zh_z'(z,t)}{h(z,t)}$ it follows by the uniform convergence in $\bar{C}_1$: $\dfrac{zh_z'(z,t)}{h(z,t)} \rightrightarrows 0$ as $t \to 1$ that for t' sufficiently close to 1 and for all $t \epsilon \langle t', 1)$ the functions $g(z,t)$ belong to S_v^0. In [3] we proved that $|h(z,t)| < 1$ in $\bar{C}_1$ for $t \epsilon \langle t_1, 1)$ and t_1 sufficiently close to 1. We have $\operatorname{Re}\{g_t'(z,t)/g(z,t)\}_{t=1} = \operatorname{Re}(1+pze^{i\theta})/(1-pze^{i\theta}) > 0$ by (E). It follows from (B), (C), (D) that

$$\arg\left\{\frac{p_0 g_z'(p_0,t)}{g_t'(p_0,t)}\right\}_{t=1} = \arg\frac{pp_0F'(pp_0)}{F(pp_0)} + \arg\frac{1+pp_0e^{i\theta}}{1-pp_0e^{i\theta}}$$

$$= \arg\frac{uF'(u)}{F(u)} + \arg\frac{1+ue^{i\theta}}{1-ue^{i\theta}} = v_0(x) - \varepsilon + 2\arctan x \,\epsilon \left(\frac{\pi}{2}, \pi\right).$$

Hence there exists $t_0 = \max(t', t_1)$ such that the homotopy $g(z,t)$ satisfies for any $t \epsilon \langle t_0, 1)$ the following conditions:

$$\text{(F)} \qquad \operatorname{Re}\left\{\frac{g_t'(z,t)}{g(z,t)}\right\} > 0 \quad \text{in } \bar{C}_1 \text{ and}$$

$$\text{(G)} \qquad \operatorname{Re}\big(g_t'(p_0,t)/p_0 g_z'(p_0,t)\big) < 0, \quad |p_0| = y.$$

Now, $v(y) < \dfrac{\pi}{2}$, hence $\left|\arg\dfrac{zg_z'(z,t)}{g(z,t)}\right| < \pi/2$ in $C_{|p_0|}$ for $t \epsilon \langle t_0, 1)$ which means that $g(z,t)$ are starlike in this disk.

By (F) and Lemma 2 we have for any $t_2, t_3 \epsilon \langle t_0, 1)$ $t_2 < t_3$: $|g(z,t_2)| < |g(z,t_3)|$ in C_1 whereas by (G) and Lemma 1 (f, F, y) does not hold which contradicts the definition of R. This proves Theorem 24.

REFERENCES

[1] Bielecki, A., Lewandowski, Z., *Sur certaines familles de fonctions a-étoilées*, Ann. Univ. Mariae Curie-Skłodowska, 15 (1961), p. 45-55.

[2] —, *Sur une généralisation de quelques théorèmes de M. Biernacki sur les fonctions analytiques*, Ann. Polon. Math. 12 (1962), pl 65-70.

[3] —, *Sur certaines majorantes des fonctions holomorphes dans cercle unité*, Coll. Math. 9 (1962), p. 299-300.

[4] —, *Sur un type de fonctions holomorphes subordonnées*, Biuletyn Lubelskiego Towarzystwa Naukowego, 2 (1962), p. 92-94.

[5] —, *A Theorem Conerning Majorants of Regular Functions*, Biuletyn Lubelskiego Towarzystwa Naukowego, 2 (1962), p. 95-96.

[6] —, *Sur un théorème concernant les fonctions univalentes linéairement accessibles de M. Biernacki*, Ann. Polon. Math. 12 (1962), p. 61-63.

[7] Biernacki, M., *Sur les fonctions univalentes*, Mathematica (Cluj) 12 (1936), p. 49-64.

[8] —, *Sur quelques majorantes de la théorie des fonctions univalentes*, C. R. Acad. Sci. Paris, 201 (1935), p. 256-258.

[9] —, *Sur la représentation conforme des domaines linéairement accessibles*, Prace Matematyczno-Fizyczne, 44 (1936), p. 293-314.

[10] Голузин, Г. М., *Мажорация подчиненных аналитических функций* I, Математический Сборник, 29 (1951), p. 209—224.

[11] —, *Геометрическая теория функций комплексного переменного*, Москва—Ленинград 1952.

[12] Kaplan, W., *Close-to-convex Schlicht Functions*, Mich. Math. Journ. 1, 2 (1952), p. 169-185.

[13] Landau, L., *Darstellung und Bergründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin 1929.

[14] Lewandowski, Z., *Sur les majorantes des fonctions holomorphes dans le cercle* $|z| < 1$, Ann. Univ. Mariae Curie-Skłodowska, 15 (1961), p. 5-11.

[15] —, *Starlike Majorants and Subordination*, Ann. Univ. Mariae Curie-Skłodowska, 15 (1961), p. 79-84.

[16] —, *Some Results Concerning Univalent Majorants*, Ann. Univ. Mariae Curie-Skłodowska, 18 (1964), p. 15-20.

[17] —, *Modular and Domain Majorants of Regular Functions*, Ann. Univ. Mariae Curie-Skłodowska, 18 (1964), p. 21-24.

[18] —, *Sur l'identité de certaines classes de fonctions univalentes II*, Ann. Univ. Mariae Curie-Skłodowska, 14 (1960), p. 19-46.

[19] Littlewood, J. E., *Lectures on the Theory of Functions*, Oxford, 1944.

[20] Löwner, K., *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann., 89 (1923), p. 103-123.

[21] Mac Gregor, T. H., *Majorization by Univalent Functions*, Duke Math. J. 34 (1967), p. 95-102.

[22] Pommerenke, Ch., *Über die Subordination analitischer Funktionen*, Journ. für die reine und andgewandte Math., 218 (1965), p. 159-173.

[23] Schaeffer, A. C., Spencer, D. C., *The Coefficients of Schlicht Functions* II, Duke Math. Journ., 12 (1945), p. 107-125.

[24] Schiffer, M., *Sur un principle nouveau pour l'evalution des fonctions holomorphe*, Bull. Soc. Math. France, 64 (1936), p. 231-240.

[25] Shah Tao-shing, *Golusin's Number* $(3-\sqrt{5})/2$ *Is the Radius of Superiority in Subordination*, Sci. Rec. 1 (1957), p. 258-261.

[26] —, *On the Radius of Superiority in Subordination*, Sci. Rec. 1, 5 (1957), p. 329-333.

[27] Strohhäcker, E., *Beitrag zur Theorie der schlichten Funktionen*, Math. Zeitschr. 37 (1933), p. 256-380.

Streszczenie

W pracy tej omawiam zagadnienia dotyczące podporządkowania funkcji i nierówności modułów, jakie w ostatnich latach były, między innymi, przedmiotem badań w lubelskim środowisku matematycznym. Praca ta zawiera też pewne nowe wyniki (twierdzenie 7, twierdzenie 24).

Резюме

В работе рассматриваются проблемы, касающиеся подчинения функций и неравенства модулей, бывшие в последние годы предметом исследований люблинских математиков. Приводятся некоторые новые результаты (теорема 7, 24).