ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLIX, 7 SECTIO A 1995

Halina HEBDA-GRABOWSKA (Lublin)

On the Rate of Convergence for Distributions of Linear Type Functionals for Sums of Infima of Independent Random Variables

ABSTRACT. Let $\{X_n\,,\ n \geq 1\}$ be a sequence of random variables uniformly distributed on $[0,1]$. Put $X_m^* = \inf(X_1, X_2, \ldots, X_m)$, $m \geq 1$; and $S_n = \sum_{m=1}^n X_m^*$, $n \geq 1$.

In this paper the convergence rate for distributions of linear functionals in the invariance principle is obtained.

1. Introduction and results. Let $\{X_n\,,\ n \geq 1\}$ be a sequence of independent random variables uniformly distributed on $[0,1]$.

Let us put

$$X_m^* = \inf(X_1, X_2, \ldots, X_m)\,,\ m \geq 1\,,$$

$$\widetilde{S}_n = \sum_{m=1}^n X_m^*,\ n \geq 1\,,\ \widetilde{S}_0 = 0\,,$$

and define

$$\text{(1)}\qquad \widetilde{S}_{n,k} = \left(S_k - \sum_{i=1}^k \frac{1}{i}\right) \Big/ \left(2 \sum_{m=1}^n \frac{1}{m}\right)^{1/2},\quad 1 \leq k \leq n\,,$$

$$\widetilde{S}_{n,0} = 0\,,\qquad n \geq 1\,.$$

We will denote by $S_n = \{S_n(t), t \in [0,1]\}$, $n \geq 1$, the random functions given as follows:

$$S_n(t) = \widetilde{S}_{n,k} + \frac{t - t_k}{t_{k+1} - t_k}\left(\widetilde{S}_{n,k+1} - \widetilde{S}_{n,k}\right),$$

(2)

$$\text{for } t \in [t_k, t_{k+1}),\ 0 \leq k \leq n-1,$$

$$S_n(0) = 0, \quad n \geq 1,$$

where $t_k = \sum_{i=1}^{k} \frac{1}{i} / \sum_{m=1}^{n} \frac{1}{m}$, $1 \leq k \leq n$, $t_0 = 0$.

Let $\{C_{[0,1]}, \mathcal{B}_C\}$ be the space of continuous functions with the uniform metric, and let $\mathcal{L}$ be a linear continuous functional defined on $C_{[0,1]}$. It is known that $\mathcal{L}$ may be written in the form

$$\mathcal{L}(x) = \int_0^1 x(t)\, d\mu(t), \tag{3}$$

where $\mu(t)$ is a function with bounded variation on $[0,1]$ (see f.e. [2], p. 304).

We are interested in finding the rate of convergence in the invariance principle:

$$\mathcal{L}(S_n) \xrightarrow{D} \mathcal{L}(W), \text{ as } n \to \infty,$$

where $W = \{W(t), t \in [0,1]\}$ is a Wiener process on $(C_{[0,1]}, \mathcal{B}_C)$.

For simplicify of notations we write

$$\Delta(X,Y) = \sup_{\lambda} |P[\mathcal{L}(X) < \lambda] - P[\mathcal{L}(Y) < \lambda]|,$$

where X, Y are random elements from $(C, \mathcal{B}_C)$.

We can now formulate the following:

Theorem. *Let $S_n = \{S_n(t), t \in [0,1]\}$, $n \geq 1$ be a sequence of random functions given by (2). Then for any linear continuous functional $\mathcal{L}$ defined on $C_{[0,1]}$ we have*

$$\Delta(S_n, W) \leq C_{\mathcal{L}} (\log n)^{-2/5}, \tag{4}$$

where $C_{\mathcal{L}}$ *is a positive constant dependent only on* $\mathcal{L}$.

This type Theorem for independent random variables has been obtained in [2].

The rate of convergence in the invariance principle for integral type functionals for sums of infima of independent random variables was investigated in [5].

2. Proof. In the proof we apply some lemmas given by P. De- héuvels ([3], [4], lemmas 3.1, 3.2) and H. Hebda–Grabowska ([5], lemmas 3.3, 3.4) which we state in Section 3 in the interests of clarity.

To shorten notation we denote $c_n = \left(2\sum_{m=1}^{n}\frac{1}{m}\right)^{1/2}$, $n \ge 1$ and set

$$V_{n,k} = \frac{[\tau_{k+1} - \tau_k - E(\tau_{k+1} - \tau_k)]}{kc_n}, \quad 1 \le k \le n$$

$$V_{n,0} = 0 , \quad n \ge 1 ,$$

where r.v's. τ_n , $n \ge 1$ are given in Section 3 by (3.1) ($\varepsilon(n) = n^{-1}$).

Put

$$U_{n,k} = \sum_{m=1}^{k} V_{n,k} , \quad 1 \le k \le n , \; n \ge 1 \tag{5}$$

and

$$L_n^{(s)} = \sum_{k=1}^{n} E|V_{n,k}|^s , \quad s \ge 2 .$$

By Lemma 3.2 r.v's. $V_{n,k}$, $1 \le k \le n$ are independent and

$$EV_{n,k} = 0 , \quad \sigma^2 V_{n,k} = \frac{2}{kc_n^2} , \quad \sigma^2 U_{n,k} = t_k ,$$

$$\sigma^2 U_{n,n} = 1 , \; 1 \le k \le n, \; n \ge 1 ,$$

moreover

$$L_n^{(s)} = O\left(s!\left(\log n\right)^{-\frac{s}{2}+1}\right), \; s \ge 2 , \; n \ge 1 . \tag{6}$$

Let us define the random functions $W_n^{(1)} = \left\{ W_n^{(l)}(t),\ t \in [0,1] \right\}$, $n \geq 1$:

$$(7)\qquad \begin{aligned} & W_n^{(1)}(t) = U_{n,k} + \frac{t - t_k}{t_{k+1} - t_k}(U_{n,k+1} - U_{n,k}) \,, \\ & W_n^{(1)}(0) = 0 \,,\ n \geq 1 \,, \end{aligned}$$

where t_k, $0 \leq k \leq n$ are as in (2).

Now using Theorem 3 ([2], p. 295) we obtain

$$(8)\qquad \Delta\left(W_n^{(1)}, W\right) \leq C_{\mathcal{L}}^{(1)} (L_n^{(s)})^{\gamma(s)} \,,$$

where $\gamma(s) = \min\left(\frac{2}{s}, \frac{1}{s-2}\right)$, $s > 2$, and $C_{\mathcal{L}}^{(1)}$ is a positive constant dependent only on the functional $\mathcal{L}$.

Adding $s = 4$ to (6) we get $L_n^{(4)} = O(4!(\log n)^{-1})$ and $\gamma(4) = 1/2$ hence from (8)

$$(9)\qquad \Delta(W_n^{(1)}, W) \leq C_{\mathcal{L}}^{(2)} (\log n)^{-1/2} \,,$$

where $C_{\mathcal{L}}^{(2)}$ depends only on $\mathcal{L}$.

Now define

$$(10)\qquad \begin{aligned} & W_n^{(2)}(t) = \widetilde{S}_{n,\tau_k} + \frac{t - t_k}{t_{k+1} - t_k}\left(\widetilde{S}_{n,\tau_{k+1}} - \widetilde{S}_{n,\tau_k}\right), \\ & \qquad\qquad t \in [t_k, t_{k+1}),\ 0 \leq k \leq n-1, \\ & W_n^{(2)}(0) = 0 \,,\ n \geq 1 \,, \end{aligned}$$

where

$$(11)\qquad \begin{aligned} & \widetilde{S}_{n,\tau_k} = \left(\sum_{i=1}^{\tau_k} X_i^* - \sum_{i=1}^{k} \frac{1}{i} \right) \Big/ c_n \,,\quad 1 \leq k \leq n \\ & \widetilde{S}_{n,\tau_0} = 0 \,,\quad n \geq 1 \,. \end{aligned}$$

By the form (3) of functional $\mathcal{L}$, (7), (10), and simply evaluations we can see that

$$\begin{aligned}
&P[|\mathcal{L}(W_n^{(2)}) - \mathcal{L}(W_n^{(1)})| \ge \delta_n] \\
&= P\left[\left|\int_0^1 \left(W_n^{(2)}(t) - W_n^{(1)}(t)\right) d\mu(t)\right| \ge \delta_n\right] \\
&= P\left[\left|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \left(W_n^{(2)}(t) - W_n^{(1)}(t)\right) d\mu(t)\right| \ge \delta_n\right] \\
&\le P\big[\max_{1\le k\le n} |U_{n,k} - \widetilde{S}_{n,\tau_k}| V_0^1\mu \ge \delta_n\big] ,
\end{aligned} \tag{12}$$

where $V_0^1\mu$ is a variation of μ on $[0,1]$.

So, by Lemma 3.3 putting $\delta_n = (\log n)^{-2/5}$ we obtain

$$P\left[\left|\mathcal{L}(W_n^{(2)}) - \mathcal{L}(W_n^{(1)})\right| \ge (\log n)^{-2/5}\right] \le C_l^{(3)} (\log n)^{-2/5} , \tag{13}$$

where $C_{\mathcal{L}}^{(3)}$ is a positive constant dependent only on $\mathcal{L}$.

By simple arguments we get

$$\begin{aligned}
\Delta\left(W_n^{(2)}, W\right) &\le \Delta\left(W_n^{(1)}, W\right) + P\left[\left|\mathcal{L}(W_n^{(2)}) - \mathcal{L}(W_n^{(1)})\right| \ge \delta_n\right] \\
&\quad + \max\Big[\sup_x |P[\mathcal{L}(W) < x + \delta_n] - P[\mathcal{L}(W) < x]|, \\
&\qquad \sup_x |P[\mathcal{L}(W) < x] - P[\mathcal{L}(W) < x - \delta_n]|\Big] .
\end{aligned} \tag{14}$$

By (9), (13) and the fact that

$$\mathcal{L}(W) = \int_0^1 W(t)\, d\mu(t)$$

is normaly distributed with zero mean and variance $\sigma^2 = \int_0^1 (\mu(1) - \mu(t))^2 dt$ ([2], p. 305) we obtain

$$\Delta(W_n^{(2)}, W) \le C_{\mathcal{L}}^{(4)} (\log n)^{-2/5} , \tag{15}$$

as

$$\sup_x |P[\mathcal{L}(W) < x \pm \delta_n] - P[\mathcal{L}(W) < x]| \le (2\pi)^{-1/2} \frac{\delta_n}{\sigma} ,$$

([6], p. 143).

Similar arguments (as in (12)), apply to the case of $S_n = \{S_n(t),\ t \in [0,1]\}$ we have

$$(16)\qquad \begin{aligned} &P\Big[\Big|\mathcal{L}(S_n) - \mathcal{L}(W_n^{(2)})\Big| \geq \delta_n\Big] \\ &\leq P\Big[\max_{1\leq k\leq n} |\widetilde{S}_{n,k} - \widetilde{S}_{n,\tau_k}| V_0^1\, \mu \geq \delta_n\Big] \\ &\leq C_{\mathcal{L}}^{(5)} (\log n)^{-2/5}\ , \end{aligned}$$

by Lemma 3.4.

Replacing in (14) $W_n^{(2)}$ by S_n, $W_n^{(1)}$ by $W_n^{(2)}$ and using (15) - (16) we get (4). This finishes the proof.

3. Lemmas. In this section we present some lemmas we needed in the proof of our Theorem.

Let $\{\varepsilon(n),\ n \geq\}$ be a sequence of positive numbers strictly decreasing to zero.

By $\{\tau_n = \tau(\varepsilon(n)),\ n \geq 1\}$ we denote the sequence of r.v's. such that

$$(3.1)\qquad \tau_n = \inf\Big\{m : \inf(X_1, \ldots, X_m) \leq \varepsilon(n)\Big\}\ ,$$

where $\{X_n\ , n \geq 1\}$ is a sequence of i.r.v's. u.d. on $[0,1]$.

Lemma 3.1. *The sequence* $\{\tau_n\ , n \geq 1\}$ *increases with probability one and* $\tau_n \to \infty$ *a.s., as* $n \to \infty$.

Lemma 3.2. *The random variables* $\tau_{n+1} - \tau_n\ , n \geq 1$ *are independent and if* $\varepsilon(n) = n^{-1}$, *then*

$$E(\tau_{n+1} - \tau_n) = 1\ , \quad \sigma^2(\tau_{n+1} - \tau_n) = 2n\ ,\ n \geq 1$$

$$\sum_{k=1}^{n} E(\tau_{k+1} - \tau_k)^p / k^p \sim p! \log n\ .$$

Lemma 3.3.

$$P\Big[\max_{1\le k\le n} |\widetilde{S}_{n,\tau_k} - U_{n,k}| \ge \delta_n\Big] \le \frac{C}{\delta_n^4 c_n^4}\,,$$

where $U_{n,k}$ *and* $\widetilde{S}_{n,\tau_k}$ *are as in* (5) *and* (11) *in* Section 2, *and* $c_n = \big(2\sum_{m=1}^{n}\frac{1}{m}\big)^{1/2}$.

Lemma 3.4.

$$P\Big[\max_{1\le k\le n} |\widetilde{S}_{n,k} - \widetilde{S}_{n,\tau_k}| > (\log n)^{-2/5}\Big] = O\Big((\log n)^{-2/5}\Big)\,,$$

where $\widetilde{S}_{n,k}$ *are given by* (1).

References

[1] Bilingsley, P., *Convergence of Probability Measures*, J. Wiley, New York, 1968.

[2] Borisov, J. S., *On the rate of convergence for distributions of integral type functionals*, Teor.Verojatn. i Primenen **21** (1976), 293–308.

[3] Dehéuvels, P., *Sur la convergence de sommes de minima de variables aléatoires*, C. R. Acad. Sci. Paris **276, A** (1973), 309–313.

[4] ———, *Valeurs extrémales d'échantillons croissants d'une variable aléatoire réelle*, Ann. Inst. H. Poincaré Probab. Statist. **10** (1974), 89–114.

[5] Hebda-Grabowska, H., *On the rate of convergence for distributions of integral type functionals for sums of infima of independent random variables*, Probab. Math. Statist. **14(2)** (1993).

[6] Petrov, V. V., *Sums of independent random variables*, Nauka, Moscow 1972.

[7] Prohorov, Yu. V., *Schodimost slucajnych processov i predelnyje theoremy veroyatnosti*, Teor. Ver. i ee Prim. **I** (1956), 177–238.

Instytut Matematyki UMCS
Plac Marii Curie–Skłodowskiej 1
20–031 Lublin, Poland

received May 8, 1995