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Inequalities for Wright-Convex Functions

Abstract. In this note we show that well-known inequalities for convex functions are also valid for Wright-convex functions.

1. Let $I \subset \mathbb{R}$ be an interval. A real function f defined on I is called Wright-convex if, for each $x < y$ and $h > 0$ such that x and $y + h$ are in I, we have

$$f(x+h) - f(x) \leq f(y+h) - f(y)\,. \tag{1}$$

Z. Opial [3] (see also [1, p. 351]) proved the following result:
If $x_{2k} \leq x_{2k-1}$, $x_{2k} \leq x_{2k+1}$ *for* $k = 1, \ldots, n$, *then*

$$\sum_{i=1}^{2n+1} (-1)^{i-1} x_i^2 \leq \left(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i \right)^2 .$$

A generalization of this inequality was given in [9] and some related results were given in [4]. The following theorem is a special case of these results:

Theorem 1. *Let $x_i \in I$, $i = 1, \ldots, 2n+1$, be such that*

$$\sum_{i=1}^{2n+1} (-1)^{i-1} x_i \in I. \tag{2}$$

(i) *If*

$$x_{2k} \le x_{2k+1}, \quad \sum_{i=1}^{2k} (-1)^{i-1} x_i \ge 0, \quad k = 1, \ldots, n \tag{3}$$

then for every convex function $f : I \to \mathbb{R}$ the following inequality holds

$$\sum_{i=1}^{2n+1} (-1)^{i-1} f(x_i) \le f\Big(\sum_{i=1}^{2n+1} (-1)^{i-1} x_i \Big). \tag{4}$$

If the reverse inequalities in (3) are valid, then (4) is also valid.

(ii) *If instead of (3) the following conditions hold*

$$x_{2k} \le x_{2k+1}, \quad \sum_{i=1}^{2k} (-1)^{i-1} x_i \le 0, \quad k = 1, 2, \ldots, n, \tag{5}$$

then the reverse inequality in (4) is valid. If the reverse inequalities in (5) are valid, then the reverse inequality in (4) is also valid.

Remark. Of course, (4) is a generalization of Opial's inequality.

Now we shall prove that Theorem 1 is also holds, if f is a Wright–convex function. In fact, we shall prove (4) only in the case when (2) and (3) are fulfilled; the other cases can be proved similarly.

Let $f : I \to \mathbb{R}$ be Wright–convex. For $n = 1$ we have to prove that

$$f(x_1) - f(x_2) + f(x_3) \le f(x_1 - x_2 + x_3) \tag{6}$$

provided that $x_2 \le x_1$ and $x_2 \le x_3$. But this is a simple consequence of (1): put $x = x_2$, $y = x_3$ and $h = x_1 - x_2$.

Suppose that Theorem 1 (with f Wright–convex) holds for $n-1$ and that (2) and (3) are fulfilled for n. Then they are also fulfilled for $n-1$, hence

$$\sum_{i=1}^{2n-1} (-1)^{i-1} f(x_i) \le f\Big(\sum_{i=1}^{2n-1} (-1)^{i-1} x_i \Big) \tag{7}$$

Since $x_{2n} \le \sum_{i=1}^{2n-1}(-1)^{i-1}x_i$ and $x_{2n} \le x_{2n+1}$, we may use (6) with x_1, x_2 and x_3 replaced by $\sum_{i=1}^{2n-1}(-1)^{i-1}x_i$, x_{2n} and x_{2n+1}, resp. It follows that

$$(8)\quad f\left(\sum_{i=1}^{2n-1}(-1)^{i-1}x_i\right) - f(x_{2n}) + f(x_{2n+1}) \le f\left(\sum_{i=1}^{2n-1}(-1)^{i-1}x_i\right).$$

From (7) and (8) we infer that Theorem 1 holds also for n; this finishes the proof.

The following theorem is a generalization of a result from [5].

Theorem 2. *Let* $y_i, z_i \in I$, $i = 1, 2, ..., n$.

(i) *If*

$$(9)\qquad z_k \le y_{k+1},\ k = 1, 2, ..., n-1,$$

$$(10)\qquad \sum_{i=1}^{k} z_i \le \sum_{i=1}^{k} y_i,\ k = 1, 2, ..., n-1,$$

$$(11)\qquad z_1 + \cdots + z_n = y_1 + \cdots + y_n.$$

Then for every Wright–convex function $f : I \to \mathbb{R}$ *the following inequality is valid:*

$$(12)\qquad f(y_1) + \cdots + f(y_n) \le f(z_1) + \cdots + f(z_n).$$

If (11) *is valid and the reverse inequalities hold in* (9) *and* (10) *then* (12) *is also valid.*

(ii) *If* (9), (11) *and the reverse inequality in* (10) *hold, then the reverse inequality in* (12) *holds. If* (10), (11) *and the reverse inequality in* (9) *hold, then the reverse inequality in* (12) *holds, too.*

Proof. Let $x_{2k-1} = y_k$, $x_{2k} = z_k$, $k = 1, ..., n$, $x_{2n+1} = z_n$. Then the assumptions of Theorem 1 are satisfied, hence (4) becomes

$$f(y_1) + \cdots + f(y_n) - f(z_1) - \cdots - f(z_n) + f(z_n) \le f(z_n).$$

Thus (12) is proved. The other cases can be proved similarly.

Remark. Assumptions of Theorems 1 and 2 are simpler than those in [7], [8].

Theorem 3. *Let* $a_1, a_{2n-1} \in I$ *and* $a_k + a_{k+1} \in I$, $k = 1, \dots, 2n-2$. *If*

$$a_1 \geq a_3 \geq \dots \geq a_{2n-1}, \tag{13}$$

$$a_{2k} \geq 0, \; k = 1, \dots, n-1, \tag{14}$$

then for every Wright–convex function $f : I \to \mathbb{R}$ *the following inequality holds*

$$\begin{aligned} f(a_1) + f(a_2 + a_3) + \dots + f(a_{2n-2} + a_{2n-1}) &\leq f(a_1 + a_2) \\ &+ \dots + f(a_{2n-3} + a_{2n-2}) + f(a_{2n-1}). \end{aligned} \tag{15}$$

If (14) *and the reverse inequalities in* (13) *are fulfilled then the reverse inequality in* (15) *is true.*

Proof. Let $y_1 = a_1$, $z_n = a_{2n-1}$, $y_{k+1} = a_{2k} + a_{2k+1}$, $z_k = a_{2k-1} + a_{2k}$, $k = 1, \dots, n-1$. Then (11) and the reverse inequalities in (9) and (10) are valid; so (15) is a consequence of (12).

2. The well-known Petrović inequality is also valid for Wright–convex functions:

Theorem 4. *Let* $f : [0, a] \to \mathbb{R}$ *be Wright–convex. If* $x_i \geq 0$, $i = 1, \dots, n$ *and* $x_1 + \dots + x_n \leq a$ *then*

$$f(x_1) + \dots + f(x_n) \leq f(x_1 + \dots + x_n) + (n-1)f(0). \tag{16}$$

Proof. For $n = 2$ (16) becomes

$$f(x_1) + f(x_2) \leq f(x_1 + x_2) + f(0). \tag{17}$$

This follows from (1) if we set $x = 0$, $y = x_1$, $h = x_2$. Suppose that Theorem 4 holds for $n-1$. Then

$$f(x_1) + \dots + f(x_{n-1}) + f(x_n) \leq f(x_1 + \dots + x_{n-1}) + f(x_n) + (n-2)f(0).$$

By using (17) with x_1 replaced by $x_1 + \dots + x_{n-1}$ and x_2 replaced by x_n we get (16); this completes the inductive proof.

3. All the above results for Wright–convex functions, as well as the Szegö inequality (see [6]), can be obtained from corresponding results for convex functions as consequences of the following result.

Theorem 5 ([2]). *Let $I \subset \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is Wright-convex if and only if $f = C + A$ where $C : I \to \mathbb{R}$ is convex and $A : \mathbb{R} \to \mathbb{R}$ is additive.*

Note also that in [2] it was proved that the well-known majorization theorem is also valid for Wright-convex functions.

Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be real numbers.

A vector $y = (y_1, \dots, y_n)$ is said to be majorized by a vector $x = (x_1, \dots, x_n)$, in symbols $x \succ y$ or $y \prec x$, if after possible reordering of the components so that $x_1 \geq \dots \geq x_n$ and $y_1 \geq \dots \geq y_n$, we have $x_1 + \cdots + x_k \geq y_1 + \cdots + y_k$, $k = 1, \dots, n-1$ and $x_1 + \cdots + x_n = y_1 + \cdots + y_n$.

Theorem 6 ([2]). *Let $I \subset \mathbb{R}$ be an open interval. The inequality*

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n) \tag{18}$$

holds for all vectors x and y with $x_i, y_i \in I$ and $x \prec y$, if and only if $f : I \to \mathbb{R}$ is Wright-convex.

Remark. It is easy to see that Theorem 4 is a consequence of Theorem 6. Also Theorem 2 is a consequence of Theorem 6 (see [5]).

Theorems 1–4 are proved in this note by using directly the definition of a Wright-convex function. The following problem could be of interest: Prove Theorem 6 by using directly the definition of a Wright-convex function.

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