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# Bounds for the hyperbolic distance in a quasidisk

Abstract. This is a survey of recent work on bounds for the hyperbolic distance $h_D$ in terms of a similarity invariant metric $j_D$ and the Möbius invariant Apollonian metric $a_D$. Both of these metrics provide lower bounds for $h_D$. Each provides an upper bound if and only if $D$ is a quasidisk.

**1. A distortion theorem.** The following surprisingly simple distortion theorem for quasiconformal mappings was established in [9].

**Theorem 1.1.** *If $f$ is a $K$-quasiconformal self mapping of $\overline{\mathbb{C}}$ which fixes 0, 1 and $\infty$, then*

$$|f(z)| + 1 \leq 16^{K-1} \left(|z| + 1\right)^K$$

*for $z \in \overline{\mathbb{C}}$. The coefficient $16^{K-1}$ cannot be replaced by a smaller constant.*

The proof follows from well known facts about the modulus of a ring domain, see for example [12], and results due to Agard [1] and Teichmüller [14].

Theorem 1.1 yields, in turn, a simple bound for the change of the cross-ratio under a quasiconformal mapping.

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**Corollary 1.2.** *If $f$ is a $K$-quasiconformal self mapping of $\overline{\mathbb{C}}$, then*

$$|(f(z_1), f(z_2), f(z_3), f(z_4))| + 1 \le 16^{K-1}(|(z_1, z_2, z_3, z_4)| + 1)^K$$

*for each quadruple of points $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$.*

**2. The distance-ratio and hyperbolic metrics.** If $h_D$ is the hyperbolic metric with curvature $-1$ in a simply connected proper subdomain $D$ of $\mathbb{C}$, then

$$j_D(z_1, z_2) \le 4\, h_D(z_1, z_2) \tag{2.1}$$

for $z_1, z_2 \in D$ where $j_D$ is the *distance-ratio metric* [7],

$$j_D(z_1, z_2) = \log\left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} + 1\right)\left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} + 1\right). \tag{2.2}$$

Since $j_D$ is a function of the ratios of the euclidean distance between $z_1$ and $z_2$ and the euclidean distances from these points to $\partial D$, $j_D$ is invariant with respect to similarities. See [8] and [11]. The function $j_D$ also yields an upper bound for $h_D$, namely

$$h_D(z_1, z_2) \le a\, j_D(z_1, z_2) + b \tag{2.3}$$

for $z_1, z_2 \in D$ if and only if $D$ is a *$K$-quasidisk*, the image of a disk or half plane under a $K$-quasiconformal self mapping of $\overline{\mathbb{C}}$; see [6]. In this case $a = a(K)$ and $b = b(K)$. Inequality (2.3) implies that

$$h_D(z_1, z_2) \le c\, j_D(z_1, z_2), \qquad c = a + \max(b, \sqrt{b}) \tag{2.4}$$

by [8]. Moreover, $h_D(z_1, z_2) \le j_D(z_1, z_2)$ if $D$ is a disk or half plane [8].

Corollary 1.2 allows us to obtain some simple estimates for $a(K)$ and $b(K)$.

**Theorem 2.5.** *If $f$ is a $K$-quasiconformal self mapping of $\mathbb{C}$, then for each proper subdomain $D$ of $\mathbb{C}$,*

$$j_{f(D)}(f(z_1), f(z_2)) \le K\, j_D(z_1, z_2) + 2(K-1)\log 16$$

*for $z_1, z_2 \in D$.*

**Corollary 2.6.** *If $D$ is a domain in $\mathbb{C}$ and if there exists a $K$-quasiconformal self mapping of $\mathbb{C}$ which maps $D$ conformally onto a disk or half plane, then*

$$h_D(z_1, z_1) \leq K\, j_D(z_1, z_2) + 2(K-1)\log 16 \tag{2.7}$$

*for $z_1, z_2 \in D$.*

Corollary 2.6 together with (2.3) and (2.4) then yield the following bounds for hyperbolic distance in a quasidisk.

**Theorem 2.8.** *If $D \subset \mathbb{C}$ is a $K$-quasidisk, then*

$$h_D(z_1, z_1) \leq K^2\, j_D(z_1, z_2) + 2(K^2-1)\log 16 \tag{2.9}$$

*for $z_1, z_2 \in D$. In particular,*

$$h_D(z_1, z_1) \leq c\, j_D(z_1, z_2)$$

*for $z_1, z_2 \in D$ where $c = c(K) \to 1$ as $K \to 1$.*

**Sketch of Proof.** By hypothesis, there exists a $K$-quasiconformal self mapping $f$ of $\overline{\mathbb{C}}$ which maps $D$ onto a disk or half plane. If $D$ is bounded, then we may assume that $f$ fixes $\infty$ and $f(D)$ is the unit disk $B$. The existence theorem for the Beltrami equation implies there exists a $K$-quasiconformal self mapping $g : B \to B$ which fixes 0 such that $g \circ f$ is conformal in $D$. Reflection in $\partial B$ extends $g$ to a $K$-quasiconformal self mapping of $\mathbb{C}$. Then $h = g \circ f$ is $K^2$-quasiconformal and we can apply Corollary 2.6 to obtain (2.9).

**Remark 2.10** The coefficient $K$ of $j_D(z_1, z_2)$ in (2.7) cannot be replaced by a constant less than $(K+1)/2$. The coefficient $K^2$ of $j_D(z_1, z_2)$ in (2.9) cannot be replaced by a constant less than $(K^2+1)/2$.

These lower bounds follow from explicit calculations for the case where $D = \{z = re^{i\theta} : 0 < r < \infty,\ |\theta| < \pi\alpha/2\}$, $0 < \alpha \leq 1$. See [9].

## 3. A sharp criterion for a quasidisk.

Ahlfors' well known three-point criterion for a quasidisk can easily be rewritten in terms of crossratios. See [6] and [8].

**Criterion 3.1.** *A Jordan domain $D$ is a quasidisk if and only if there is a constant $c \geq 1$ such that*

$$|(z_1, z_4, z_2, z_3)| + |(z_3, z_4, z_2, z_1)| = \frac{|z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|}{|z_1 - z_3||z_2 - z_4|} \leq c$$

*for each ordered quadruple of points $z_1, z_2, z_3, z_4$ in $\partial D$.*

How large can the constant $c$ be for a $K$-quasidisk? For this we recall the distortion function

$$\lambda(K) = \left( \frac{\phi_K(1/\sqrt{2})}{\phi_{1/K}(1/\sqrt{2})} \right)^2 \quad \text{where} \quad \phi_K(r) = \mu^{-1}(\mu(r)/K).$$

See [12], [3]. Here $K > 0$, $0 < r < 1$ and $\mu(r)$ is the modulus of the ring domain bounded by $\{z : |z| = 1\}$ and the segment $[0, r]$.

The function $\lambda(K)$ gives the sharp upper bound for the distortion of the unit circle,

$$\sup_{\theta_1, \theta_2} \left( \frac{|f(e^{i\theta_1}) - f(0)|}{|f(e^{i\theta_2}) - f(0)|} \right),$$

under a $K$-quasiconformal self map $f$ of $\overline{\mathbb{C}}$; see [13].

Theorem 2 of [2] then yields the following result.

**Criterion 3.2.** *If $z_1, z_2, z_3, z_4$ is an ordered quadruple of points on a $K$-quasicircle $C$, then*

$$\frac{|z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|}{|z_1 - z_3||z_2 - z_4|} \leq c(K) \tag{3.3}$$

*where*

$$c(K) = \frac{\lambda(K)^{1/2} + \lambda(K)^{-1/2}}{2}.$$

*Inequality (3.3) is sharp for all $K$.*

Additional calculation then yields the following variant of Criterion 3.2 [10].

**Criterion 3.4.** *If $z_1, z_2, z_3, z_4$ is an ordered quadruple of points on a $K$-quasicircle $C$ and if*

$$\max(|z_1 - z_0|, |z_3 - z_0|) \leq a \leq b \leq \min(|z_2 - z_0|, |z_4 - z_0|)$$

*for some point* $z_0 \in \mathbb{C}$, *then*

$$b/a \leq \lambda(K)^{1/2}. \tag{3.5}$$

*Inequality* (3.5) *is sharp for all* $K$.

**4. The Apollonian and hyperbolic metrics.** If $D$ is a proper subdomain of the extended complex plane $\overline{\mathbb{C}}$, then the *Apollonian metric* $a_D$ is defined as

$$\begin{aligned} a_D(z_1, z_2) &= \sup_{w_1, w_2 \in \partial D} \log |(z_1, z_2, w_1, w_2)| \\ &= \sup_{w_1, w_2 \in \partial D} \log \left( \frac{|z_1 - w_1||z_2 - w_2|}{|z_1 - w_2||z_2 - w_1|} \right) \end{aligned} \tag{4.1}$$

for $z_1, z_2 \in D$. See [4] and [5]. Strictly speaking $a_D$ is only a pseudometric if $\partial D$ lies in a circle $C$ and $D$ contains points which are symmetric in $C$. It follows that $a_D$ is invariant with respect to Möbius transformations.

The metric $a_D$ furnishes information about other metrics defined in $D$. For example, if $D$ is a disk, then $a_D(z_1, z_2) = h_D(z_1, z_2)$ for $z_1, z_2 \in D$ [Be]. If $D \subset \overline{\mathbb{C}}$ is a simply connected domain of hyperbolic type, then $a_D$ yields the sharp lower bound

$$a_D(z_1, z_2) \leq 2\, h_D(z_1, z_2) \tag{4.2}$$

for $h_D$ [5]. Cf. (2.1). Finally it follows from (2.2) and (4.1) that

$$a_D(z_1, z_2) \leq j_D(z_1, z_2) \tag{4.3}$$

even when $D$ is not simply connected [Be].

As in the case of the distance-ratio metric $j_D$, the Apollonian metric $a_D$ yields an upper bound for $h_D$ only when $D$ is a quasidisk. This fact is a consequence of the following analogues for $a_D$ of inequalities (2.7) and (2.9). The geometric information concerning $\partial D$ in Criterion 3.4 is needed in the proof of these results. For example, as noted above, there exist $z_1, z_2 \in D$ with $a_D(z_1, z_2) = 0 < h_D(z_1, z_2)$ whenever $\partial D$ is a proper subset of a circle.

**Theorem 4.4.** *If* $f$ *is a* $K$*-quasiconformal self map of* $\overline{\mathbb{C}}$ *and if* $D$ *is a proper subdomain of* $\overline{\mathbb{C}}$, *then*

$$a_{f(D)}(f(z_1), f(z_2)) \leq K\, a_D(z_1, z_2) + 2(K-1)\log 32$$

*for* $z_1, z_2 \in D$.

**Corollary 4.5.** *If $D$ is a domain in $\overline{\mathbb{C}}$ and if there exists a $K$-quasiconformal self map $f$ of $\overline{\mathbb{C}}$ which maps $D$ conformally onto a disk, then*

$$h_D(z_1, z_2) \le K\, a_D(z_1, z_2) + 2(K-1)\log 32. \tag{4.6}$$

**Theorem 4.7.** *If $D \subset \overline{\mathbb{C}}$ is a $K$-quasidisk, then*

$$h_D(z_1, z_2) \le K^2\, a_D(z_1, z_2) + 2(K^2-1)\log 32 \tag{4.8}$$

*for $z_1, z_2 \in D$. In particular, $h_D(z_1, z_2) \le c\, a_D(z_1, z_2)$ where $c = c(K) \to 1$ as $K \to 1$.*

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