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Invo-regular unital rings

Abstract. It was asked by Nicholson (Comm. Algebra, 1999) whether or not unit-regular rings are themselves strongly clean. Although they are clean as proved by Camillo–Khurana (Comm. Algebra, 2001), recently Nielsen and Šter showed in Trans. Amer. Math. Soc., 2018 that there exists a unit-regular ring which is not strongly clean. However, we define here a proper subclass of rings of the class of unit-regular rings, called *invo-regular* rings, and establish that they are strongly clean. Interestingly, without any concrete indications a priori, these rings are manifestly even commutative invo-clean as defined by the author in Commun. Korean Math. Soc., 2017.

1. Introduction and background. Everywhere in the text of the present article, all rings are assumed to be associative, containing the identity element 1 which, in general, differs from the zero element 0. Our terminology and notations are mainly in agreement with those from [7] and [10]. For instance, for such a ring R, the symbol $U(R)$ stands for the group of units, $Inv(R)$ for the set of all involutions (= square roots of 1), $Id(R)$ for the set of all idempotents and $Nil(R)$ for the set of all nilpotents.

In [6] there was introduced the following important notion.

Definition 1.1. A ring R is said to be *unit-regular* if, for each $r \in R$, there is $u \in U(R)$ such that $r = rur$.

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Since the equality $r = rur$ is always tantamount to the condition that ur is an idempotent, say e, it is apparent that $r = u^{-1}e$, and conversely.

The following also appeared in [6].

Definition 1.2. A ring R is said to be *strongly regular* if, for each $r \in R$, there is $u \in U(R)$ such that $r = r^2u$, i.e., $r = rur$ with $ur = ru$.

It is well known that all strongly regular rings are unit-regular (see, for instance, [6] or [7]), but the converse fails in general. In fact, all elements r in strongly regular rings can be written also like this $r = wf = fw$ for some $w \in U(R)$ and $f \in Id(R)$.

On the other hand, in [12] was defined the following famous concept.

Definition 1.3. A ring R is called *clean* if, for every $r \in R$, there are $u \in U(R)$ and $e \in Id(R)$ with $r = u + e$. If, in addition, the commutativity condition $ue = eu$ is satisfied, the clean ring R is said to be *strongly-clean*.

There was shown in [1] the fundamental fact that unit-regular rings are necessarily clean. Furthermore, Nicholson (cf. [13]) asked if a unit-regular ring is even strongly clean. Recently, this was answered in [14] in the negative by constructing a very special unit-regular ring which is, in fact, not strongly clean.

The aim of the current paper is to take into account the specific nature of this example from [14] and to prove that there are quite natural proper subclasses of unit-regular rings which are strongly clean. This will be done in the subsequent section. We terminate the paper with some left-open problems.

2. Main definition and result. We first begin with our key instruments.

Definition 2.1. ([3]) A ring R is called *strongly invo-regular* if, for each $r \in R$, there exists $v \in Inv(R)$ such that $r = r^2v$, that is, $r^2 = rv$.

It was obtained in [3, Theorem 2.2] that a ring R is strongly invo-regular if, and only if, $R \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ, μ.

In order to get a worthwhile generalization of these rings, it is also of some interest to consider those rings R for which either $r^2 - rv \in Nil(R)$ or $r^2 - rv \in Id(R)$. However, we will proceed in another way, namely we shall now slightly extend Definition 2.1 to the next one as follows:

Definition 2.2. A ring R is said to be *invo-regular* if, for every $r \in R$, there exists $v \in Inv(R)$ such that $r = rvr$, that is, $r = ve$, where $e = vr \in Id(R)$.

It is straightforward to see that a ring R is invo-regular exactly when R is unit-regular and $U(R) = Inv(R)$.

Recall that, imitating [4], a ring R is called *invo-clean* if any $r \in R$ can be represented as $r = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$.

Proposition 2.3. *Invo-regular rings are invo-clean.*

Proof. Since invo-regular rings are unit-regular, we appeal to [1] to get that they are clean. However, as already commented above, all units being involutions imply that invo-regular rings have to be invo-clean, as claimed. □

We now need some more technicalities.

Lemma 2.4. *In any invo-regular ring* $6 = 0$.

Proof. Letting $2 \in R$ for some invo-regular ring R, it must be that $2 = ve$ for some $v \in Inv(R)$ and $e \in Id(R)$. Thus $2v = e$ and squaring we have $4 = e = 2v$. Again squaring, we obtain that $16 = 4$, i.e., $12 = 0$. Hence $6^2 = 3.12 = 0$ and so $6 \in Nil(R)$. Since the Jacobson radical of unit-regular rings is always zero (see, e.g., [7]), this enables us that $6 \in J(R) = \{0\}$, as expected. □

Proposition 2.5. *A ring R is invo-regular if, and only if, $R \cong R_1 \times R_2$, where R_1 is an invo-regular ring of characteristic* 2 *and R_2 is an invo-regular ring of characteristic* 3.

Proof. Since $(2, 3) = 1$ and by Lemma 2.4 we have $6 = 0$, a standard trick works to deduce that $R = 2R \oplus 3R$. Hence $R/2R \cong 3R$ and $R/3R \cong 2R$ which both imply that $R \cong R_1 \times R_2$, where $R_1 = R/2R$ and $R_2 = R/3R$. But it is not too hard to check that homomorphic images of an invo-regular ring are again invo-regular rings, which gives our claim. □

We now arrive at the following central result as, surprisingly, the following holds.

Theorem 2.6. *The following three items are equivalent:*
(i) *R is invo-regular.*
(ii) *R is strongly invo-regular.*
(iii) *R is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. The equivalence (ii) $\iff$ (iii) was proved in [3]. That is why we will deal now only with the implication (i) $\Rightarrow$ (iii). To that goal, with Proposition 2.5 at hand, one can write $R \cong R_1 \times R_2$, where both R_1, R_2 are invo-regular and $char(R_1) = 2$ whereas $char(R_2) = 3$.

In the first case, because of the equality $U(R_1) = Inv(R_1)$, it follows that $U(R_1) = 1 + Nil(R_1)$ as $(1 + Inv(R_1))^2 = 1 + Inv^2(R_1) = 0$. Thus R_1 is a UU ring (i.e. all units are unipotent units). However, it was proved in [2], and independently in [5], that unit-regular UU rings are Boolean. Consequently, R_1 must be Boolean, as asserted.

In the second case, Proposition 2.3 tells us that R_2 is an invo-clean ring of characteristic 3, whence [4, Theorem 2.15] allows us to conclude that R_2 can be embedded in a direct product of copies of the field $\mathbb{Z}_3$. □

Remark 2.7. Paralleling a similar argumentation to that from [5], one can say something more as follows: If point (iii) is true, then R is a commutative regular ring and every unit of R is an involution, so that the implications "(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)" follow immediately. As for the remaining one "(i) $\Rightarrow$ (iii)", we process thus: For any $a \in R$, write $a = ev$, $e^2 = e$ and $v^2 = 1$. Then $a^3 = a$ in case e is central. So, it suffices to prove that R is reduced whence it will be necessarily abelian. In fact, if R has a non-zero, square-zero element a, then by a well-known classical result in ring theory, mainly attributed to Levitzki (see, e.g., [5]), R will contain a corner subring eRe isomorphic to a full 2 by 2 matrix ring over a non-trivial ring. Thus, owing to the specific nature of units in matrix rings, it is plainly checked that eRe contains a unit u whose square is not e and, therefore, $u + 1 - e$ is a unit of R that is not an involution observing that $(u+1-e)^2 = u^2+1-e$ (compare with Proposition 2.16, too). This is, however, the wanted contradiction.

We continue with some uniqueness in the class of invo-regular rings, defined like this:

Definition 2.8. A ring R is said to be *uniquely invo-regular* if, for every $0 \neq r \in R$, there is a unique $v \in Inv(R)$ such that $r = rvr$.

Since as we have seen above r can be written as $r = ve$, where $e = vr \in Id(R)$, a question which immediately arises is whether or not the uniqueness of v is retained in this record as well. Specifically, the following simple but useful assertion is true.

Proposition 2.9. *The ring R is uniquely invo-regular if, and only if, for each $0 \neq r \in R$, there is a unique $v \in Inv(R)$ with $r = ve$ for some $e \in Id(R)$.*

Proof. "$\Rightarrow$". Writing $r = ve = wf$ for some $v, w \in Inv(R)$ and $e, f \in Id(R)$, it follows that $vr = e$ and $wr = f$. So, $vrvr = vr$ and $wrwr = wr$, that is, $rvr = r = rwr$. This yields that $v = w$ (whence $e = f$), as required.

"$\Leftarrow$". Letting $r = rvr = rwr$ for some $v, w \in Inv(R)$, it follows that $r = ve = wf$ for $e = vr \in Id(R)$ and $f = wr \in Id(R)$. This implies that $v = w$ (and hence $e = f$), as needed. □

We are now ready to establish the following more comprehensive result, by using a technique developed in [8].

Theorem 2.10. *A ring R is uniquely invo-regular if, and only if, either $R \cong B$, where B is a Boolean ring, or $R \cong \mathbb{Z}_3$.*

Proof. The sufficiency being trivial, we concentrate on the necessity. We claim that if R is uniquely invo-regular, then R is either a Boolean ring or a division ring. First of all, we will detect that all elements of R are either idempotents or involutions. And so, for an arbitrary $a \in R$ we write $ava = a$ for some unique v with $v^2 = 1$, and assume that $a^2 \neq a$ (i.e., $v \neq 1$). We will show that $a^2 = 1$. In fact, notice that all units in R are involutions as well as we can represent the element a as follows:

$$a[v(1-a(1-av))]a = ava = a[(1-(1-va)a)v]a,$$

where $(1-a(1-av))^{-1} = 1+a(1-av)$ and $(1-(1-va)a)^{-1} = 1+(1-va)a$ because $a(1-va)a = a(1-av)a = 0$. The uniqueness now yields that

$$v(1-a(1-av)) = v = (1-(1-va)a)v,$$

that is,

$$a(1-av) = 0 = (1-va)a.$$

The last equalities ensure that $a^2v = a = va^2$, i.e., $av = a^2 = va$. This means that $a^3 = a$ and hence $a^4 = a^2$. Therefore, one may verify that $(a+1-a^2)^2 = 1$ and $(1-a^2)^2 = 1-a^2$. But the relationships

$$a(a+1-a^2)a = a = ava$$

lead us to $v = a+1-a^2$ whence $v(1-a^2) = (a+1-a^2)(1-a^2) = 1-a^2$. If now $a^2 \neq 1$, by what we have just shown above, we infer that

$$(1-a^2)v(1-a^2) = 1-a^2 = (1-a^2).1.(1-a^2).$$

The uniqueness assures that $v = 1$, a contradiction. This substantiates that $a^2 = 1$, as promised.

Next, to argue the initial claim, suppose R is not Boolean. So, there is $b \in R$ with $b^2 \neq b$. By what we have already established, $b^2 = 1$. Let $x \in R \setminus \{0\}$ and consider bx.

Case 1: $(bx)^2 = bx$. Thus $bxbx = bx$ gives $xbx = x$. But $x^2 \neq x$ as otherwise $x = x.1.x$ and so the uniqueness forces $b = 1$ and hence $b^2 = b$, contrary to our assumption. Finally, in view of our conclusions above, $x^2 = 1$.

Case 2: $(bx)^2 \neq bx$. So $(bx)^2 = 1$, which means that $bxbx = 1$, that is, $xbx = b$. Therefore, since b is invertible with the inverse b being an involution, it readily follows that x is invertible with the inverse bxb. This finalizes our claim at the beginning of the proof that R is a division ring.

Furthermore, utilizing Theorem 2.6 alluded to above, let us assume that R is not Boolean. Thus, it follows from the first part above that R is a division ring and, resultantly, being indecomposable, R is a subdirect product of the $\mathbb{Z}_3$'s. It is easily seen that each element $y \in R$ now satisfies the equation $y^3 = y$. If $y \neq 0$, then y inverts and so $y^2 = 1$, which amounts to $(y-1)(y+1) = 0$. If $y \neq 1$, we have $y = -1$ as well as if $y \neq -1$, we have $y = 1$. This guarantees that $R \cong \mathbb{Z}_3$, as stated. □

Remark 2.11. Certainly, utilizing Theorem 2.6, the proof of the previous theorem can be simplified like this: It is quickly observed that $(1-av)av = 0$ since $ava = a$, and hence we deduce $(1-av)a = 0$, implying that $(1-va)a = 0$ because R is a commutative ring being invo-regular.

Moreover, a sketch of a parallel proof of the preceding theorem, by the usage of another arguments, could be the following one: One only needs to show the necessity. By part (iii) of Theorem 2.6, it follows that $x^3 = x$ for every $x \in R$. So x^2 is an idempotent for each $x \in R$. Since the case $2 = 0$ obviously leads us to the obtaining of a Boolean ring, one can assume that $char(R) = 3$ or, in other words, that R is a subdirect product of isomorphic copies of the field $\mathbb{Z}_3$. Thus, to prove that $R \cong \mathbb{Z}_3$, it suffices to establish that R has no non-trivial idempotents. In doing that, let $e \neq 0, 1$ be an arbitrary idempotent. Then one verifies that the equalities $-e = e(1-2e)e = e(-1)e$ hold, where $1 - 2e = 1 + e$ and -1 are distinct involutions as $3 = 0$. This is a contradiction, however.

Mimicking the same idea for proof as that from the preceding theorem, we can find a necessary and sufficient condition when for the strongly regular ring R the next additional condition is valid: For any $r \in R$ there is a unique $v \in Inv(R)$ such that $r = r^2v$. The expected result, which can be proved in a way of similarity to that quoted above, will be again either the Boolean ring or the three element field.

Another way to consider uniqueness is the following one:

Definition 2.12. A ring R is called *pseudo uniquely invo-regular* if, for each $r \in R$, there is a unique $e \in Id(R)$ with $r = ve$ for some $v \in Inv(R)$.

So, we have now at our disposal all the ingredients necessary to prove the following rather surprising statement.

Theorem 2.13. *A ring R is pseudo uniquely invo-regular if, and only if, it is invo-regular.*

Proof. One way being trivial, we concentrate on the other one. To this aim, given R is an invo-regular ring. Appealing to Theorem 2.6, accomplishing it with [9], we deduce that R is a commutative ring and all elements y in R satisfy the equation $y^3 = y$ which amounts to $y = ve$, where $v = -y^2 + y + 1 \in Inv(R)$ and $e = y^2 \in Id(R)$. Supposing $y = ve = wf$, for some $w \in Inv(R)$ and $f \in Id(R)$, with the commutativity in mind we detect that $y^2 = v^2e^2 = w^2f^2$, i.e., $y^2 = e = f$, as expected. □

The following note is helpful.

Remark 2.14. Notice that in the above decomposition $y = ve = wf$, although we concluded $e = f$, it could be that $v \neq w$ as the next example unambiguously illustrates: Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ and consider the element $(-1, 0)$. It can be twicer written as $(-1, 0) = (-1, 1)(1, 0) = (-1, -1)(1, 0)$, so that by substituting $v = (-1, 1) \neq (-1, -1) = w$ and $e = (1, 0) = f$ the claim sustained.

We continue with certain element-wise properties of (strongly) invo-regular elements. Imitating Definition 2.1 (see cf. [3]), an element a from a ring R is called *strongly invo-regular* if there is $v \in Inv(R)$ such that $a = a^2v = va^2$. A natural question, regarding this element a, is whether or not it can be presented as a strongly invo-clean element (e.g., [4]), which is of the form $a = w + f$, where $w \in Inv(R)$ and $f \in Id(R)$ with $wf = fw$. The answer is the positive "yes" and is subsumed by the following.

Proposition 2.15. *Any strongly invo-regular element is strongly invo-clean.*

Proof. It follows directly from the definition that $a = a^2v = va^2$ and $va = av = a^2$ for some $v \in Inv(R)$. Writing $a = ev = ve$, where $e = a^2 = e^2$, one observes that $a = (ve + e - 1) + (1 - e)$. What remains to show is that $w = ve + e - 1$ is an involution, because $f = 1 - e$ is always an idempotent which commutes with w. Indeed, squaring $w^2 = (ve)^2 + e + 1 + 2ve^2 - 2ve - 2e$ and taking into account that $(ve)^2 = v^2e^2 = e$ and $e^2 = e$, we obtain the wanted equality that $w^2 = 1$. □

We will now examine how invo-regularity is situated in the corner rings. It is well known that (see, e.g., [11]) that if R is a unit-regular ring, then the corner subring eRe is also unit-regular for any $e \in Id(R)$. We will now prove the following similarity.

Proposition 2.16. *If R is an invo-regular ring and $e \in Id(R)$, then eRe is an invo-regular ring, too.*

Proof. By the comments above, eRe is unit-regular. But we shall show that $U(eRe) = Inv(eRe)$ which is enough to conclude that eRe is invo-regular. To that purpose, letting $u \in U(eRe)$, we obtain that $u + 1 - e \in U(R)$ with the inverse $u' + 1 - e$, where $uu' = u'u = e$. However, $u(1-e) = (1-e)u = 0$ and so $1 = (u + 1 - e)^2 = u^2 + (1 - e)^2 = u^2 + 1 - e$. This insures that $u^2 = e$, as required. □

We finish off our work with three queries of interest and importance.

Problem 2.17. *If R is a ring and $e \in Id(R)$ such that $eRe \cong (1-e)R(1-e) \cong \mathbb{Z}_2$, is it true that R is unit-regular?*

For some arbitrary but a fixed $n \in \mathbb{N}$, we will say that a ring R is *n-torsion regular* if, for any r, there exists $w \in U(R)$ with $w^n = 1$, such that $r = rwr$. It is self-evident that n-torsion regular rings are of necessity unit-regular with bounded by n unit groups.

Problem 2.18. *Does it follow that n-torsion regular rings are strongly clean?*

It is worthwhile noticing that if R is a p-torsion regular ring of prime $char(R) = p$, then R has to be Boolean. Indeed, for every $u \in U(R)$ we have $u^p = 1$ and so $u = 1 + (u - 1) \in 1 + Nil(R)$ bearing in mind that $(u - 1)^p = u^p - 1 = 0$. This allows us to derive that R is a unit-regular UU ring and, as we already have seen above, the application of [2] or [5] enables us that R is necessarily Boolean, as pursued.

And so, we close the queries with the following element-wise question.

Problem 2.19. *Is an invo-regular element in an arbitrary ring a clean element in this ring?*

In fact, it can be written for such an element r in a ring R that $r = ev = vf$ for some $e, f \in Id(R)$ and $v \in Inv(R)$ with $v(e+f) = (e+f)v$. It is worth noticing that this question has a negative resolution for unit-regular elements (cf. [14]). So, for a possible counterexample to this query, one could look at the ring $R = \mathbb{M}_2(\mathbb{Z})$ which is manifestly not clean as the ring of integers $\mathbb{Z}$ is not clean.

Now, for an arbitrary but fixed $n \in \mathbb{N}$, we will say that a ring R is *strongly n-torsion regular* if, for any r, there exists $w \in U(R)$ with $w^n = 1$, such that $r = r^2 w$. It is elementarily seen that $r = r^{n+1}$ and hence, owing to the famous theorem of Jacobson, strongly n-torsion regular rings are of necessity commutative. So, we come to

Problem 2.20. *Characterize up to an isomorphism strongly n-torsion regular rings.*

As a final challenging problem, we state:

Problem 2.21. *Characterize those rings R for which, for each $r \in R$, there exists $e \in Id(R) \cap [Inv(R)r]$ such that $r - re = r(1-e) \in Nil(R)$.*

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