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A survey of a selection of methods for determination of Koebe sets

Abstract. In this article we take over methods for determination of Koebe set based on extremal sets for a given class of functions.

1. Introduction. Let $\mathcal{A}$ denote a set of all functions that are analytic in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ such that every $f \in \mathcal{A}$ satisfies the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote a class of functions $f \in \mathcal{A}$ such that the functions f are univalent in Δ.

Definition 1. We define the Koebe set of the class A, where $\mathrm{A} \subset \mathcal{A}$ is the point-set $\bigcap_{f\in \mathrm{A}} f(\Delta)$ and denote it by $\mathcal{K}(\mathrm{A})$, so we have

$$\mathcal{K}(\mathrm{A}) = \bigcap_{f\in \mathrm{A}} f(\Delta).$$

The set $\mathcal{K}(\mathrm{A})$ is a "maximal" set such that for every function $f \in \mathrm{A}$ the set $\mathcal{K}(\mathrm{A}) \subset f(\Delta)$, i.e. if for the set B we have that $\mathrm{B} \subset f(\Delta)$ for every function $f \in \mathrm{A}$, then $\mathrm{B} \subset \mathcal{K}(\mathrm{A})$.

Definition 2. Let m_{A} be an analytic and univalent function in the unit disk Δ. The function m_{A} is called a minorant of the class A if the set $m_{\mathrm{A}}(\Delta)$ is the maximal set such that $m_{\mathrm{A}}(\Delta) \subset f(\Delta)$ for every function f from the class A provided that this function exists.

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From the definition of minorant we have that the minorant of the class A exists if the Koebe set $\mathcal{K}(\mathrm{A})$ is a domain.

Remark 1. There are classes of functions for which the Koebe set is not a domain. For example, for the class $\mathcal{T}_{a_3}$ of typically-real functions with the fixed third coefficient $a_3 = \frac{f'''(0)}{3!}$ the set $\mathcal{K}(\mathcal{T}_{a_3})$ is a collection of three disconnected domains, where

$$\mathcal{T}_{a_3} := \left\{ f \in \mathcal{A} : \mathfrak{Im}\, z\, \mathfrak{Im}\, f(z) \geq 0 \text{ for } z \in \Delta \text{ and } \frac{f'''(0)}{3!} = a_3 \right\},$$

$a_3 \in [-1, 3]$.

The determination of the Koebe set for the class $\mathcal{T}_{a_3}$ is complicated and this problem has been considered in [3].

We can give some other examples of classes of univalent functions for which the set $\mathcal{K}(\mathrm{A})$ is a collection of three disconnected domains, for example $\mathrm{A} = \{f(z), -f(-z)\}$, where $f \in \mathcal{S}$ and

$$f(\Delta) = \mathbb{C} \setminus (\{\omega = \omega_0 + t, t \geq 0\} \cup \{\omega = \overline{\omega}_0 + t, t \geq 0\})$$

where $\mathfrak{Re}\, \omega_0 < 0$ and $\mathfrak{Im}\, \omega > 0$. Hence, we see that the Koebe set does not have to be bounded.

2. Examples of Koebe sets.

1. The Koebe set for the class $\mathcal{T}$ of typically-real functions.

The Koebe set for the class $\mathcal{T} := \{f \in \mathcal{A} : \mathfrak{Im}\, f(z)\, \mathfrak{Im}\, z \geq 0,\ z \in \Delta\}$ of typically-real functions was founded by A. W. Goodman [1] in 1977. The set $K(\mathcal{T})$ is symmetric with respect to the real axis, and its boundary in the upper half plane is a curve given by the polar equation

$$g(\theta) = \begin{cases} \frac{1}{4}, & \text{if } \theta = 0 \text{ or } \theta = \pi, \\ \frac{\pi \sin\theta}{4\theta(\pi - \theta)}, & \text{if } 0 < \theta < \pi. \end{cases}$$

In the proof of this fact Goodman used the universal function $F(z) = \frac{1}{\pi} \tan \frac{\pi z}{1+z^2}$ for which $F(\Delta) = \mathbb{C} \setminus \left\{-\frac{i}{\pi}, \frac{i}{\pi}\right\}$ and $\pm\frac{i}{\pi} \in \partial\mathcal{K}(\mathcal{T})$.

From the fact that $F_c(z) := \frac{F(\frac{z+c}{1+cz}) - F(c)}{(1-c^2)F'(c)}$ belongs to the class $\mathcal{T}$ for $c \in (-1, 1)$, we have

$$\frac{\pm\frac{i}{\pi} - F(c)}{(1-c^2)F'(c)} \in \partial\mathcal{K}(\mathcal{T}).$$

This means that the boundary in the upper half plane of the domain $\mathcal{K}(\mathcal{T})$ is given by the parametric equation

$$\omega(c) = \begin{cases} \frac{\frac{i}{\pi} - F(c)}{(1-c^2)F'(c)} & \text{for } c \in (-1, 1), \\ -\frac{1}{4} & \text{for } c = -1, \\ \frac{1}{4} & \text{for } c = 1. \end{cases}$$

From this we can get the polar equation.

2. The Koebe set of one subclass of the class of all functions that are convex in the direction of the imaginary axis.

A function f is convex in the direction of $e^{i\alpha}$ if f maps the unit disk Δ onto a domain convex in the direction of $e^{i\alpha}$. This means that each line parallel to a given line with the direction of $e^{i\alpha}$ either misses $f(\Delta)$ or is contained in $f(\Delta)$ or the intersection with $f(\Delta)$ is either a segment or a ray. Functions of this class will be denoted by $\mathcal{CV}(e^{i\alpha})$.

For the class $\mathcal{Q} \subset \mathcal{H}$ we define

$$\mathcal{QR} := \{f \in \mathcal{Q} : a_n \in \mathbb{R} \text{ for } n \in \mathbb{N}_0\}.$$

Let $\mathcal{CVR}(i)$ be the class of all functions that are convex in the direction of the imaginary axis. We have $f \in \mathcal{CVR}(i)$ if and only if for every $\omega \in \partial f(\Delta)$,

$$\mathfrak{Im}\,\omega > 0 \Rightarrow (f(\Delta) \cap \{\omega + it, t \geq 0\} = \emptyset \ \wedge \ f(\Delta) \cap \{\overline{\omega} + it, t \leq 0\} = \emptyset).$$

Using this property, we can consider the subclass of the class $\mathcal{CVR}(i)$.

Let for a fixed α from the interval $[0,1]$

$$K_{\omega,\alpha} := \left\{z : (1-\alpha)\frac{\pi}{2} \leq \arg(z-\omega) \leq (1+\alpha)\frac{\pi}{2}, \text{ where } \omega \in \mathbb{C}\right\}$$

and

$$A_{\omega,\alpha} := \mathbb{C} \setminus \left\{K_{\omega,\alpha} \cup \overline{K}_{\omega,\alpha}\right\}, \text{ where } \overline{K}_{\omega,\alpha} := \{\overline{\omega} : \omega \in K_{\omega,\alpha}\}.$$

Definition 3. $f \in \mathcal{CVR}_\alpha(i)$ if and only if

$$\underset{\omega \in \partial f(\Delta)}{\forall} \ \mathfrak{Im}\,\omega \geq 0 \Rightarrow \left(f(\Delta) \cap K_{\omega,\alpha} = \emptyset \ \wedge \ f(\Delta) \cap \overline{K}_{\omega,\alpha} = \emptyset\right).$$

It is easy to see that for $\alpha_1 < \alpha_2$ we have $\mathcal{CVR}_{\alpha_2}(i) \subset \mathcal{CVR}_{\alpha_1}(i)$. The class $\mathcal{CVR}_\alpha(i)$ is convex in the direction of $e^{i\theta}$ for $\theta \in [(1-\alpha)\frac{\pi}{2}, (1+\alpha)\frac{\pi}{2}]$.

The set $A_{\omega,\alpha}$ is the domain for $\omega \neq 0$ and $\mathfrak{Im}\,\omega > 0$. For $\mathfrak{Im}\,\omega > 0$ from the Riemann theorem we have that there exists only one univalent function $f_{\omega,\alpha}$ in the unit disk Δ such that $f(\Delta) = A_{\omega,\alpha}$, $f_{\omega,\alpha}(0) = 0$ and $f'_{\omega,\alpha}(0) > 0$.

Let $\mathcal{K}(A)$ be a domain and the point $\omega \in \partial\mathcal{K}(A)$.

Definition 4. The function $f_\omega \in A$ such that $\omega \in \partial f_\omega(\Delta)$ is called the extremal function for a given Koebe domain for the class A and the domain $f_\omega(\Delta)$ is called the extremal domain for the class A.

Theorem 5. *If $\mathfrak{Im}\,\omega > 0$, then the set $A_{\omega,\alpha}$ is the extremal domain for the class $\mathcal{CVR}_\alpha(i)$ when $f'_{\omega,\alpha}(0) = 1$.*

Proof. Let $\mathfrak{Im}\,\omega > 0$ and $f'_\omega(0) = 1$. From the definition of the class $\mathcal{CVR}_\alpha(i)$ we have that the function $f_{\omega,\alpha} \in \mathcal{CVR}_\alpha(i)$. Assume that there exists a function $f \in \mathcal{CVR}_\alpha(i)$ such that the point $\omega - \varepsilon i \in \partial f(\Delta)$ for ε with $0 < \varepsilon \leq \mathfrak{Im}\,\omega$. By the definition of the domain $A_{\omega,\alpha}$ we have $A_{\omega,\alpha-\varepsilon i} \subset A_{\omega,\alpha}$ and by the definition of the class $\mathcal{CVR}_\alpha(i)$ we have $f(\Delta) \subset A_{\omega-\varepsilon i}$. Hence,

$f(\Delta) \subset f_{\omega-\varepsilon i}(\Delta) \subsetneq f_\omega(\Delta)$, which means that $f \prec f_{\omega-\varepsilon i}$ and $f_{\omega-\varepsilon i} \prec f_\omega$. Hence, $1 = f'(0) \leq f'_{\omega-\varepsilon i}(0) < f'_\omega(0) = 1$, which is a contradiction. Hence, the interval $[\mathfrak{Re}\,\omega, \omega) \subset f(\Delta)$ for every function f from the class $\mathcal{CVR}_\alpha(i)$.

Due to real coefficients the segment $(\overline{\omega}, \omega) \subset f(\Delta)$ for every function f from the class $\mathcal{CVR}_\alpha(i)$, so we have $(\overline{\omega}, \omega) \subset \mathcal{K}(\mathcal{CVR}_\alpha(i))$. From this and the fact that $\omega \in \partial f_\omega(\Delta)$ we have that $\omega \in \partial\mathcal{K}(\mathcal{CVR}_\alpha(i))$ and $\omega \in \partial\mathcal{K}(\mathcal{CVR}_\alpha(i))$ also when $\omega \in \mathbb{R}$. $\square$

From the Schwarz–Christoffel formula we have

$$f_{\omega,\alpha}(z) = \int_0^z \frac{\left[(\zeta - e^{i\theta})(\zeta - e^{-i\theta})\right]^{1-\alpha}}{(1-\zeta^2)^{2-\alpha}} d\zeta,$$

where

$$\omega = \omega(\theta) = \int_0^{e^{i\theta}} \frac{\left[(\zeta - e^{i\theta})(\zeta - e^{-i\theta})\right]^{1-\alpha}}{(1-\zeta^2)^{2-\alpha}} d\zeta, \quad \theta \in [0, \pi].$$

From the above, we have

Theorem 6. *The Koebe set of the class $\mathcal{CVR}_\alpha(i)$ is a domain and its boundary is a curve given by the equation*

$$\omega(\theta) = \int_0^1 \frac{e^{i\theta}\left[(1-t)(1-te^{2i\theta})\right]^{1-\alpha}}{(1-t^2e^{2i\theta})^{2-\alpha}} dt, \quad \theta \in [-\pi, \pi],$$

where $\omega(\theta)$ for $\theta \in [-\pi, 0]$ determines the equality $\omega(\theta) = \overline{\omega(-\theta)}$.

3. Other forms of the Koebe domains for the class $\mathcal{CVR}_\alpha(i)$.

(1) Notice that the Bieberbach's transformation $\frac{f(\frac{z+c}{1+cz}) - f(c)}{(1-c^2)f'(c)}$ remains invariant in $\mathcal{CVR}_\alpha(i)$ and the extremal functions $f_{\omega(\theta)}$ for $c \in (-1, 1)$. Moreover, for $\mathfrak{Im}\,\omega(\theta) > 0$ we have

$$\{f_{\theta,c} : c \in (-1,1)\} = \left\{f_{\omega(\theta)} : \theta \in (0, \pi)\right\},$$

where

$$f_{\theta,c}(z) := \frac{f_{\omega(\theta)}(\frac{z+c}{1+cz}) - f_{\omega(\theta)}(c)}{(1-c^2)f'_{\omega(\theta)}(c)}.$$

Taking $\theta = \frac{\pi}{2}$, we have

$$\frac{\omega(\frac{\pi}{2}) - \int_0^c \frac{(1+\zeta^2)^{1-\alpha}}{(1-\zeta^2)^{2-\alpha}} d\zeta}{(1-c^2)^{\alpha-1}(1+c^2)^{1-\alpha}} \in \partial\mathcal{K}(\mathcal{CVR}_\alpha(i)).$$

It means that the boundary of Koebe domain for the class $\mathcal{CVR}_\alpha(i)$ is given by the equation

$$\upsilon(c) = \left(\frac{1-c^2}{1+c^2}\right)^{1-\alpha} \left(\omega\left(\frac{\pi}{2}\right) - \int\limits_0^c \frac{(1+\zeta^2)^{1-\alpha}}{(1-\zeta^2)^{2-\alpha}} d\zeta\right).$$

(2) A minorant of the class $\mathcal{CVR}_\alpha(i)$.
By Theorem 2, we have the equation of boundary of the domain for the class $\mathcal{K}(\mathcal{CVR}_\alpha(i))$

$$\omega(\theta) = \int\limits_0^1 \frac{e^{i\theta}\left[(1-t)(1-te^{2i\theta})\right]^{1-\alpha}}{(1-t^2e^{2i\theta})^{2-\alpha}} dt, \quad \theta \in [-\pi, \pi].$$

Notice that for the function

$$f(z) := \int\limits_0^1 \frac{z\left[(1-t)(1-t^2z^2)\right]^{1-\alpha}}{(1-t^2z^2)^{2-\alpha}} dt$$

we have $f(e^{i\theta}) = \omega(\theta)$ for $\theta \in [-\pi, \pi]$. Hence, $f(\Delta) = \mathcal{K}(\mathcal{CVR}_\alpha(i))$, which means that $\frac{1}{f'(0)} f(z) \in \mathcal{CVR}_\alpha(i)$. From the above, we see that the minorant of the class $\mathcal{CVR}_\alpha(i)$ is the function $f(z)$, therefore $m_{\mathcal{CVR}_\alpha(i)}(z) = f(z)$. Hence, $\mathcal{K}(\mathcal{CVR}_\alpha(i)) = f(\Delta)$, where

$$f(z) = \int\limits_0^1 \frac{z\left[(1-t)(1-t^2z^2)\right]^{1-\alpha}}{(1-t^2z^2)^{2-\alpha}} dt.$$

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