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On zeros of Bloch functions and related spaces of analytic functions

Dedicated to Professor Zdzisław Lewandowski on his 70th birthday

Abstract. In this paper we consider some problems for zero sets of Bloch functions and A^p functions. We improve some necessary conditions for ordered zero sequences and show that they are best possible.

1. Introduction. Let A^p, $0 < p < \infty$, denote the Bergman space of functions f analytic in the unit disc $\mathbb{D}$ satisfying

$$\|f\|_p = \left(\frac{1}{\pi}\iint_{\mathbb{D}} |f(z)|^p \, dx \, dy\right)^{1/p} < \infty$$

A function f analytic in $\mathbb{D}$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty .$$

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The space of all Bloch functions will be denoted by $\mathcal{B}$. The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ for which

$$(1-|z|)|f'(z)| \to 0, \quad \text{as } |z| \to 1.$$

For $0<r<1$, set

$$M_\infty(r,f) = \max_{|z|=r} |f(z)|.$$

and let us define A^0 as the space of all functions f analytic in $\mathbb{D}$ and such that

$$M_\infty(r,f) = \mathrm{O}\left(\log \frac{1}{1-r}\right), \quad \text{as } r \to 1.$$

The following inclusions are well known

$$\mathcal{B}_0 \subset \mathcal{B} \subset A^0 \subset \bigcap_{0<p<\infty} A^p.$$

If f is an analytic function in $\mathbb{D}$, $f(0) \neq 0$, and $\{z_k\}_{k=1}^\infty$ is the sequence of its zeros, repeated according to multiplicity and ordered so that $|z_1| \leq |z_2| \leq |z_3| \ldots$, then $\{z_k\}$ is said to be the sequence of ordered zeros of f. In 1974 Horowitz [H1] proved that if $\{z_k\}$ is the sequence of ordered zeros of $f \in A^p$, then

$$\prod_{k=1}^{N} \frac{1}{|z_k|} = \mathrm{O}\left(N^{1/p}\right), \quad \text{as } N \to \infty. \tag{1}$$

The result in [GNW, Theorem 1] shows that $\mathrm{O}\left(N^{1/p}\right)$ can be replaced by $\mathrm{o}\left(N^{1/p}\right)$.

Moreover, Horowitz [H1] obtained the following necessary condition for ordered zeros of A^p functions

Theorem H. *Assume that $f \in A^p$, $0<p<\infty$, $\{z_k\}$ is the ordered zero set of f and $b_k = 1-|z_k|$. Then for all $\varepsilon > 0$*

$$\sum_{b_k \neq 1} b_k \left(\log\left(\frac{1}{b_k}\right)\right)^{-1-\varepsilon} < \infty.$$

Horowitz also proved that this result is best possible in the sense that the series

$$\sum_{b_k \neq 1} b_k \left(\log\left(\frac{1}{b_k}\right)\right)^{-1}$$

may diverge for $f \in A^p$. The analogous result for the space A^0 was obtained in [GNW], that is, if $f \in A^0$ and b_k are as in Theorem H, then for all $\varepsilon > 0$

$$\sum_{|z_k|>1-\frac{1}{e}} b_k \left(\log\log\left(\frac{1}{b_k}\right)\right)^{-1-\varepsilon} < \infty.$$

This result is also best possible in the sense that there exists a function $f \in \mathcal{B}_0$ for which

$$\sum_{|z_k|>1-\frac{1}{e}} b_k \left(\log\log\left(\frac{1}{b_k}\right)\right)^{-1} = \infty.$$

In Section 1 of this paper we improve the above mentioned necessary conditions for ordered zeros of A^p functions and A^0 functions.

It follows from (1) that, if $f \in A^p$ and $\{z_k\}$ are the ordered zeros of f, then

$$\limsup_{n\to\infty} \frac{n(1-|z_n|)}{\log n} \le \frac{1}{p} .$$

E. Beller in [B] proved that the constant $\frac{1}{p}$ is best possible. Namely, for given $\varepsilon > 0$ he constructed a function $f \in A^p$ such that

$$\limsup_{n\to\infty} \frac{n(1-|z_n|)}{\log n} > \frac{1}{p(1+\varepsilon)} .$$

For A^0 space and for the Bloch space, by Theorem 2 in [GNW], we get the analogous inequalities

$$\limsup_{n\to\infty} \frac{n(1-|z_n|)}{\log\log n} \le 1 \quad \text{if } f \in A^0 , \tag{2}$$

$$\limsup_{n\to\infty} \frac{n(1-|z_n|)}{\log\log n} \le \frac{1}{2} \quad \text{if } f \in \mathcal{B} . \tag{3}$$

Theorem 2 in [GNW] does not imply the sharpness of (2) and (3). It does not even exclude the possibility that lim sup in (2) and (3) are always 0. Actually, for the function $f \in A^0$ constructed in the proof of Theorem 3 [GNW] we have $\limsup_{n\to\infty} \frac{n(1-|z_n|)}{\log\log n} = 0$ In Section 2 we give an example of the function $f \in A^0$ for which $\limsup_{n\to\infty} \frac{n(1-|z_n|)}{\log\log n} \ge \frac{1}{4}$. The sharpness of (2) and (3) is still an open problem.

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2. Necessary conditions for zeros of A^p functions. Define $\log_1 x = \log x$, $\log_n x = \log(\log_{n-1} x)$, for $n = 2, 3\ldots$ and sufficiently large x. For a given positive integer n let r_n denote the solution of the equation $\log_{n-1} x = 0$. We have the following

Theorem 1. *Let $f \in A^p$ and $\{z_k\}$ be the ordered zero set of f. Let $b_k = 1 - |z_k|$. Then for all positive integers n and all $\varepsilon > 0$*

$$\sum_{|z_k|>r_n} \frac{b_k}{\log\left(\frac{1}{b_k}\right)\log\log\left(\frac{1}{b_k}\right)\ldots\log_{n-1}\left(\frac{1}{b_k}\right)\left(\log_n\left(\frac{1}{b_k}\right)\right)^{1+\varepsilon}} < \infty.$$

Proof. We will apply the arguments analogous to that in the proof of Theorem H [H1, p.697]. In this proof and in what follows C denotes a positive constant which may be different at each occurrence. Condition (1) implies that for N sufficiently large

$$Nb_N \le \sum_{k=1}^{N} b_k \le C\log(N+1)$$

and consequently,

$$\frac{1}{\log_i\left(\frac{1}{b_N}\right)} \le \frac{1}{C\log_i(N+1)}\ , \quad 1 \le i \le n-1,$$

and

$$\frac{1}{\left[\log_n\left(\frac{1}{b_N}\right)\right]^{1+\varepsilon}} \le \frac{1}{c_n\left[\log_n(N+1)\right]^{1+\varepsilon}}\ .$$

Multiplying the above inequalities we get

$$\frac{1}{\prod_{i=1}^{n-1}\log_i\left(\frac{1}{b_N}\right)\left[\log_n\left(\frac{1}{b_N}\right)\right]^{1+\varepsilon}} \le \frac{1}{C\prod_{i=1}^{n-1}\log_i(N+1)\left[\log_n(N+1)\right]^{1+\varepsilon}}\ .$$

Hence it suffices to show that

$$\sum_{k=k_0}^{\infty} \frac{b_k}{\prod_{i=1}^{n-1}\log_i(k+1)\left[\log_n(k+1)\right]^{1+\varepsilon}} < \infty.$$

Set

$$\phi_n(x) = \left(\prod_{i=1}^{n-1}\log_i x\right)^{-1}(\log_n x)^{-1-\varepsilon}\ .$$

Then

$$\phi_n'(x) = -\frac{\prod_{i=2}^{n}\log_i x + \prod_{i=3}^{n}\log_i x + \cdots + \prod_{i=n-1}^{n}\log_i x + \log_n x + 1 + \varepsilon}{x\left(\prod_{i=1}^{n-1}\log_i x\right)^2(\log_n x)^{2+\varepsilon}}\ .$$

By the mean value theorem, $\phi_n(k+2) - \phi_n(k+1) = \phi_n'(x_0)$, where $x_0 \in [k+1, k+2]$. Hence

$$\begin{aligned}
&\phi_n(k+1) - \phi_n(k+2) \\
&= \frac{\prod_{i=2}^{n} \log_i x_0 + \prod_{i=3}^{n} \log_i x_0 + \cdots + \prod_{i=n-1}^{n} \log_i x_0 + \log_n x_0 + 1 + \varepsilon}{x_0 \left(\prod_{i=1}^{n-1} \log_i x_0\right)^2 (\log_n x_0)^{2+\varepsilon}} \\
&< \frac{n \prod_{i=2}^{n} \log_i x_0}{x_0 \left(\prod_{i=1}^{n-1} \log_i x_0\right)^2 (\log_n x_0)^{2+\varepsilon}} \\
&= \frac{n}{x_0 (\log x_0)^2 \left(\prod_{i=2}^{n-1} \log_i x_0\right) (\log_n x_0)^{1+\varepsilon}} \\
&< \frac{n}{(k+1)\left(\log(k+1)\right)^2 \left(\prod_{i=2}^{n-1} \log_i (k+1)\right) (\log_n (k+1))^{1+\varepsilon}} .
\end{aligned}$$

Summing by parts gives

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \frac{b_k}{\prod_{i=1}^{n-1} \log_i (k+1) \left[\log_n (k+1)\right]^{1+\varepsilon}} \\
&\le \lim_{k\to\infty} \frac{C \log(k+1)}{\prod_{i=1}^{n-1} \log_i (k+1) \left[\log_n (k+1)\right]^{1+\varepsilon}} \\
&+ C \sum_{k=k_0}^{\infty} \frac{\log(k+1)}{(k+1)\left(\log(k+1)\right)^2 \left(\prod_{i=2}^{n-1} \log_i (k+1)\right) (\log_n (k+1))^{1+\varepsilon}} \\
&= C \sum_{k=k_0}^{\infty} \frac{1}{(k+1) \left(\prod_{i=1}^{n-1} \log_i (k+1)\right) (\log_n (k+1))^{1+\varepsilon}} < \infty. \quad \square
\end{aligned}$$

Remark 1. In [H1] Horowitz constructed a function $f \in A^p$ whose zeros satisfy the inequality

$$b_k \ge \frac{c}{k}$$

where $c > 0$ is independent of k. For this function we have

$$\sum_{|z_k| > r_n} \frac{b_k}{\log\left(\frac{1}{b_k}\right) \log\log\left(\frac{1}{b_k}\right) \ldots \log_{n-1}\left(\frac{1}{b_k}\right) \log_n\left(\frac{1}{b_k}\right)} = \infty.$$

This means that Theorem 1 is best possible in the sense that $\varepsilon > 0$ cannot be omitted.

Using the result in [Theorem 2, GNW] and the reasoning as in the proof of the preceding theorem one can get

Theorem 2. *Let $f \in A^0$ and $\{z_k\}$ be the ordered zero set of f. Let $b_k = 1 - |z_k|$. Then for all positive integers n and all $\varepsilon > 0$*

$$\sum_{|z_k|>r_n} \frac{b_k}{\log\log\left(\frac{1}{b_k}\right)\dots\log_{n-1}\left(\frac{1}{b_k}\right)\left(\log_n\left(\frac{1}{b_k}\right)\right)^{1+\varepsilon}} < \infty.$$

Remark 2. The result of Theorem 2 is best possible in the sense that there exists a function $f \in \mathcal{B}_0$ for which

$$\sum_{|z_k|>r_n} \frac{b_k}{\log\log\left(\frac{1}{b_k}\right)\dots\log_n\left(\frac{1}{b_k}\right)} = \infty. \tag{4}$$

Proof. For an analytic function f in $\mathbb{D}$ such that $f(0) \neq 0$, let $n(r,f)$ denote the number of zeros of f in the disc $\{|z| \leq r\}$, where each zero is counted according to its multiplicity. We define also

$$N(r,f) = \int_0^r \frac{n(t,f)}{t}dt.$$

It was proved in [O], (see also [GNW]) that there exists $f \in \mathcal{B}_0$, $f(0) \neq 0$, such that for some $\beta > 0$

$$N(r,f) \geq \beta \log\log\frac{1}{1-r}, \quad r_0 < r < 1. \tag{5}$$

Let $\{z_n\}$ be the ordered sequence of zeros of f. We will show that for such a function f, (4) holds. For simplicity, set $n(r,f) = n(r)$ and $N(r,f) = N(r)$. Integrating by parts we obtain

$$\sum_{|z_k|>r_n} \frac{b_k}{\log\log\left(\frac{1}{b_k}\right)\dots\log_n\left(\frac{1}{b_k}\right)} \geq \int_{r_0}^1 \frac{1-r}{\prod_{i=2}^n \log_i \frac{1}{1-r}}dn(r) + \mathrm{O}(1)$$

$$= \int_{r_0}^1 r\left[\prod_{i=2}^n \log_i\left(\frac{1}{1-r}\right)\right]^{-1}\left(1+\frac{1+\sum_{j=3}^n\prod_{i=j}^n \log_i\left(\frac{1}{1-r}\right)}{\prod_{i=1}^n \log_i\left(\frac{1}{1-r}\right)}\right)\frac{n(r)}{r}dr+\mathrm{O}(1)$$

$$\geq \int_{r_0}^1 r\left[\prod_{i=2}^n \log_i\left(\frac{1}{1-r}\right)\right]^{-1}\frac{n(r)}{r}dr + \mathrm{O}(1).$$

Another integration by parts and (5) give

$$\begin{aligned}
&\int_{r_0}^{1} r\left[\prod_{i=2}^{n}\log_i\left(\frac{1}{1-r}\right)\right]^{-1}\frac{n(r)}{r}dr\\
=&\int_{r_0}^{1}\left(-\left[\prod_{i=2}^{n}\log_i\left(\frac{1}{1-r}\right)\right]^{-1}+\frac{r\left(1+\sum_{j=3}^{n}\prod_{i=j}^{n}\log_i\left(\frac{1}{1-r}\right)\right)}{(1-r)\log\left(\frac{1}{1-r}\right)\left[\prod_{i=2}^{n}\log_i\left(\frac{1}{1-r}\right)\right]^2}\right)\\
&\times N(r)dr+\mathrm{O}(1)\\
\geq&\int_{r_0}^{1}\left(-\left[\prod_{i=2}^{n}\log_i\left(\frac{1}{1-r}\right)\right]^{-1}+\frac{r\left(\prod_{i=3}^{n}\log_i\left(\frac{1}{1-r}\right)\right)}{(1-r)\log\left(\frac{1}{1-r}\right)\left[\prod_{i=2}^{n}\log_i\left(\frac{1}{1-r}\right)\right]^2}\right)\\
&\times N(r)dr+\mathrm{O}(1)\\
=&\int_{r_0}^{1}\left(-\left[\prod_{i=2}^{n}\log_i\left(\frac{1}{1-r}\right)\right]^{-1}+\frac{r}{(1-r)\log_2\left(\frac{1}{1-r}\right)\prod_{i=1}^{n}\log_i\left(\frac{1}{1-r}\right)}\right)\\
&\times N(r)dr+\mathrm{O}(1)\\
\geq& C\int_{r_0}^{1}\frac{r}{1-r}\frac{1}{\prod_{i=1}^{n}\log_i\left(\frac{1}{1-r}\right)}dr+\mathrm{O}(1)=\infty,
\end{aligned}$$

which ends the proof of Remark 2. $\square$

3. Zeros of A^0 functions. If $f \in A^0$ and $\{z_k\}$ are the ordered zeros of f, then by Theorem 2 in [GNW]

$$\prod_{k=1}^{n}\frac{1}{|z_k|}\leq c\log n.$$

Since $\{|z_n|\}$ is nondecreasing, we have

$$n(1-|z_n|)\leq\sum_{k=1}^{n}(1-|z_k|)<\sum_{k=1}^{n}-\log|z_k|\leq\log c+\log\log n,$$

which implies (2).

Now we prove

Theorem 3. *There exists $f \in A^0$ such that*

$$\limsup_{n\to\infty}\frac{n(1-|z_n|)}{\log\log n}\geq\frac{1}{4},$$

where $\{z_n\}$ are the ordered zeros of f.

Proof. The reasoning we are going to apply in our proof is related to that used by Horowitz [H2,p. 330]. Let $n_k = 2^{2^{2^k}}$. Set

$$F_k(z) = \frac{1 + 2^{2^{k-2}} z^{n_k - n_{k-1}}}{1 + 2^{-2^{k-2}} z^{n_k - n_{k-1}}}$$

and

$$f(z) = \prod_{k=1}^{\infty} F_k(z)\,.$$

For every k, the function F_k has exactly $n_k - n_{k-1}$ zeros on the circle

$$|z| = 2^{-\frac{2^{k-2}}{n_k - n_{k-1}}} = e^{-\frac{2^{k-2}\log 2}{n_k - n_{k-1}}}.$$

Moreover, we have

$$\begin{aligned}\frac{n_k(1-|z_{n_k}|)}{\log\log n_k} &= \frac{n_k\left(1 - e^{-\frac{2^{k-2}\log 2}{n_k - n_{k-1}}}\right)}{\log\log n_k} \\ &= \frac{n_k\left(\frac{2^k \log 2}{4(n_k - n_{k-1})} - \frac{1}{2}\left(\frac{2^k\log 2}{4(n_k - n_{k-1})}\right)^2 + \dots\right)}{2^k \log 2 + \log\log 2} \to \frac{1}{4} \quad , k \to \infty\,,\end{aligned}$$

since $\frac{n_{k-1}}{n_k} \to 0\,,$ as $k \to \infty$. Hence

$$\limsup_{n\to\infty} \frac{n(1-|z_n|)}{\log\log n} \geq \frac{1}{4}\,.$$

Now we will prove that $f \in A^0$.
If $|z| = r_N = 2^{-1/n_N}$, then

$$|f(z)| = \left|\prod_{k=1}^{N} F_k(z)\right| \left|\prod_{k=N+1}^{\infty} F_k(z)\right|$$

and

$$\left|\prod_{k=1}^{N} F_k(z)\right| = \prod_{k=1}^{N} 2^{2^{k-2}} \left|\frac{2^{-2^{k-2}} + z^{n_k - n_{k-1}}}{1 + 2^{-2^{k-2}} z^{n_k - n_{k-1}}}\right| < 2^{2^{-1} + 2^0 + \dots + 2^{N-2}} < 2^{2^{N-1}}.$$

Since $n_k - n_{k-1} > \frac{1}{2} n_k$ and $g(x) = \frac{a+x}{1+ax}$ is increasing, $a \in (0,1)$, for $|z| = r_N$ we get

$$\left| \prod_{k=N+1}^{\infty} F_k(z) \right| = \prod_{k=N+1}^{\infty} 2^{2^{k-2}} \left| \frac{2^{-2^{k-2}} + z^{n_k - n_{k-1}}}{1 + 2^{-2^{k-2}} z^{n_k - n_{k-1}}} \right|$$

$$\leq \prod_{k=N+1}^{\infty} 2^{2^{k-2}} \frac{2^{-2^{k-2}} + |z|^{n_k - n_{k-1}}}{1 + 2^{-2^{k-2}} |z|^{n_k - n_{k-1}}}$$

$$< \prod_{k=N+1}^{\infty} \frac{1 + 2^{2^{k-2}} 2^{-\frac{1}{2} \frac{n_k}{n_N}}}{1 + 2^{-2^{k-2}} 2^{-\frac{1}{2} \frac{n_k}{n_N}}} \; .$$

It suffices to show that

$$\sum_{k=N+1}^{\infty} 2^{-(\frac{1}{2} \frac{n_k}{n_N} - 2^{k-2})} < C,$$

where C is independent of N. To this end, put

$$p_k = 2^{2^{2^k} - 2^{2^N} - 1} - 2^{k-2}.$$

For $k > N$, $\{p_k\}$ is an increasing subsequence of positive integers, so

$$\sum_{k=N+1}^{\infty} 2^{-p_k} < \sum_{k=N+1}^{\infty} 2^{-k} = 2^{-N} < 1\,.$$

Hence $|f(z)| < C 2^{2^N}$, if $|z| = r_N$. Note that

$$r_N = e^{-\frac{\log 2}{2^{2^{2^N}}}} = 1 - \frac{\log 2}{2^{2^{2^N}}} + \frac{1}{2} \left(\frac{\log 2}{2^{2^{2^N}}} \right)^2 - \dots$$

and consequently,

$$\log \frac{1}{1 - |z|} \sim \log \frac{2^{2^{2^N}}}{\log 2} \sim 2^{2^N}.$$

This means that

$$|f(z)| = O\left(\log \frac{1}{1 - |z|} \right) \;, \; |z| = r_N.$$

Now, if $r_N \leq |z| \leq r_{N+1}$ we get

$$|f(z)| \leq M_\infty(r_{N+1}, f) \leq 2^{2^N} \leq C \log \frac{1}{1 - r_N} \leq C \log \frac{1}{1 - |z|}.$$

Hence $f \in A^0$. This finishes the proof. $\square$

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